THE USE OF AN ELECTRONIC ANALOG COMPUTER IN THE DETERMINATION OF THE NORMAL MODES OF LATERAL VIBRATION OF NON-UNIFORM BEAMS

THE USE OF AN ELECTRONIC ANALOG COMPUTER [illegible]

IN THE [illegible] OF THE [illegible]

OF INITIAL VELOCITY [illegible]

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SUMMARY

It is the purpose of this paper to describe an investigation of the normal modes of lateral vibration of a free-free beam having variable stiffness and mass considering the effects of bending and shear deflections and rotary inertia, by means of an electronic system. A practical engineering problem of this type is found in the vibrations of an aircraft wing, or, as in the case of this investigation, the normal modes of vibrations of a naval vessel for which data and solutions by other methods were readily accessible.

The components of the electronic system used consist of standard feedback amplifier units whose interconnections are wires, resistors, and capacitors, relay switches, and associated power supplies.

Results obtained considering bending deflection only are in close agreement with results obtained by graphical methods and an IBM computer. Results obtained considering additional effects of shear deflection and rotary inertia have limited accuracy due to the assumptions made in setting up the computer equation.

This investigation was conducted in the laboratories of the Aeronautical Engineering Department,

University of Michigan, [illegible] of the requirements

for Master Degrees, by [illegible] and [illegible] S. Parrott.

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INTRODUCTION

Solutions of dynamic problems in fields other than electrical have been obtained by electrical analogs for some time.[1,2] The early use of amplifiers in the electrical analog networks added complications due to variation of tube characteristics. However, with the advent of the stabilized feed-back amplifier new techniques were developed.[3,4] With these new techniques multi-unit computers have been designed in compact form where the elements are interconnected by plugging into jacks at a common patch bay. One such computer is the Reeves Electronic Analog Computer developed by the Reeves Instrument Corporation for the Special Devices Center of the Office of Naval Research.[5]

The techniques employed herein are based on individual stabilized feedback amplifier units designed after the circuits given in Ref. 4, by C. E. Howe and R. Howe. for their investigations with electronic computers conducted at the University of

Michigan during the summer of 1950.[6] The report of their investigations has not yet been published. The equipment developed during the above investigation was made available to the authors for their use in this project.

This equipment consists of plug-in feedback amplifier units which by simple external changes in connections will serve as integrator, differentiator, multiplier or divider, and sign changer. Because of this ability to perform the operations of arithmetic and calculus on an input voltage the units are termed "operational amplifiers." Provided the associated power supplies are well regulated and the circuit components are of high precision, the accuracy of the results of these operations is adequate for most engineering problems. The components of the computer other than the operational amplifiers and their associated power supplies consist of wires, resistors, capacitors, and relay switches for use in setting initial conditions and introducing the variables of the problems into the system. Solutions in the form of output voltages are amplified and recorded on a paper tape recorder.

The Howes' investigation[6] included solutions of problems dealing with deflections and normal modes of vibration of beams for various loading and end conditions. A solution of the normal modes of vibration of a free-free beam with variable moments of inertia and variable mass was obtained considering bending deflections only. The example used was the hull of a fictitious ship whose moment of inertia and mass distributions were symmetrical about its middle section.

The primary problem considered herein is an extension of the above in that moments of inertia and mass distributions are not symmetrical about the middle section and the effects of shear deflections and rotary inertia are included. Data for the problem were taken from an actual vessel, the APA [illegible], of the United States Navy.

An investigation of the normal modes of vibration of the same vessel by analytical methods (Stodola) has been conducted by the University of Michigan, Engineering Mechanics Department, in conjunction with work under contract for the United States Navy.[7] Solutions were obtained for the first and second normal modes considering bending deflections only. Under another United States Navy contract to the International Business Machines Corporation, solutions of the first five normal modes considering bending deflections only, and also with additional effects of shear deflections and rotary inertia were obtained by means of a mechanical-electrical computing machine.[8] Results obtained by the above two methods are compared with those obtained in the following investigation which uses a relatively simple electronic analog computer.

Several simpler practical problems dealing with beams are first solved in order to familiarize the authors with the use of the equipment and to demonstrate the simplicity of computer set-ups for increasing complicity of the differential equations to be solved. These problems are followed by the solutions of the first three normal modes of vibration of the APA 87 considering bending deflections only, and of the first four normal modes considering the additional effects of shear deflections and rotary inertia.

The authors wish to express their appreciation to Associate Professors R. J. Lesher, M. H. Nichols, and L. L. Rauch, Aeronautical Engineering Department, University of Michigan, for their assistance, cooperation, and guidance in carrying on this investigation.

EQUIPMENT AND PROCEDURE

As stated in the introduction the various elements necessary for setting up a complete computer circuit were previously assembled and available as laboratory equipment. A complete description of the elements are to be found in Ref. [illegible]. Since this reference is at present time unpublished, brief descriptions of the various components used are given herein.

The unit upon which the computer is based is a direct current, three-stage, feedback amplifier having good stability, high gain and an effective phase shift of 180 degrees. In the following problems the amplifier is used as a multipler or divider, sign changer, and integrator. To have the amplifier perform any of these operations it is necessary only to make external changes of the impedances in the input and feedback circuits. Fig. 1 shows the circuit diagram of the operational amplifier. Fig. 2 is a photograph of the chassis in which the circuit is mounted. Fig. 3 and Fig. 4 are photographs of the operational amplifier as multiplier and integrator respectively. Due to the 180 degrees phase shift in the amplifier it changes the sign of the input voltage in every operation. For pure sign changing the input and feedback impedances consist of equal resistors; for multiplication the desired ratios of feedback to input resistors are plugged into the amplifier circuit.

The amplifiers are connected to a power supply distribution panel by means of six-wire shielded cables. The two knobs shown on the chassis of the amplifier are used to balance the amplifier for zero direct current output prior to use in the computer circuit. The knobs are connected to two variable resistors associated with the input tube.

To balance a multiplying or sign changing amplifier, the input is shorted to ground and equal resistors (one megohm) are plugged into the input and feedback circuits. By use of a multi-range direct current vacuum tube voltmeter connected between ground and the output, zero output is obtained by adjusting the two knobs.

The procedure for balancing the integrating amplifier is similar. With a one megohm resistor and a one mocrofarad capacitor as input and feedback impedances respectively, balance is obtained by adjusting the knobs for a constant output.

The procedure for balancing combinations of amplifiers in a computer set-up will differ somewhat depending on the problem or combination. A small unbalance in an individual amplifier may build up, particularly through integrating amplifiers, into unacceptable over-all unbalance of the computer. A procedure for balancing a complete computer circuit of six amplifiers is described later in the problem dealing with the vibrations of a ship's hull.

The jacks seen in the chassis of the amplifier were designed for banana plugs and spaced for General Radio Type 274-M double plugs. Input and feedback resistors are Continental X-type, accurate to plus or minus one percent. Polystyrene capacitors used in the feedback circuits of integrating amplifiers are Western Electric, one microfarad capacitors having high leakage resistance and low dielectric absorption.

The high voltage power supplies required for the amplifiers consist of plus and minus 350 and minus 150 well regulated and filtered direct current voltages Fig. 1. These voltages together with a six volt supply for the amplifier heaters, are taken to a distribution box by means of shielded cables. The distribution box used has twelve outlets for the connections of amplifiers. Although both alternating current voltage and direct current voltage supplies were available at the distribution box for the amplifier heater circuit, the latter was found to reduce the amount of sixty cycle oscillations which were especially noticeable in the computer outputs when several integrating amplifiers were in use.

Due to the low output impedance of the operational amplifiers the output current is small necessitating a direct current power amplifier with a high ratio of input to output impedances in order to obtain satisfactory records of the computer outputs. For this purpose Brush, Model BL-913, direct current amplifiers were placed between the desired outputs of the computer and the recording oscillograph. The recording oscillograph used was a two channel, Brush, Model BL-202, magnetic type.

The equipment used for simulating the variable coefficients of the equations and setting the initial conditions of the problems consists of a synchronous contactor, units of stepping relay switches with their panels for lug-in resistors, initial condition relay switches, and a relay control panel.

Fig. 5 shows a photograph of a stepping relay unit with panel having the proper lug-in resistor assemblies for simulating the variable moments of inertia along the length of a ship's hull used in a later discussion. The stepping relays are driven by electric pulses from a synchronous motor and switch assembly. The synchronous motor drives a shaft at one revolution per second. On the shaft are mounted two cams, one having one flat, the other having four flats. An Und. Lab., Inc., type BZ-2RL2 microswitch rides on each of the cams; the one giving four pulses per second drives the stepping relays; the switch giving one pulse per second is sent directly to the relay control panel for the purpose of starting the computer problems always on the same flat of the stepping relay cam. This uniformity is provided to minimize the effects of the machining tolerances of the cam flats on the timing of the problem.

The stepping relay circuit diagram is shown in Fig. 6. Each relay consists of three arcs of forty contacts each. Two of the contact arcs have bridging wipers while the third has a non-bridging wiper. The latter is used to impress and remove initial conditions while one of the former is used to cut in the series connected resistors on the plug-in panel. With this equipment it is therefore possible to simulate a variable function, of a beam for instance, in forty stations along the beam. The number of stations used together with the number of relay contacts wiped per second determine the length of the problem beam in terms of seconds. To obtain a correct solution to a problem then, the imposed end conditions must be satisfied within the exact length of the beam as determined by the stepping relay panel set-up and motor-driven cam combination. Such a combination is described later for the varying properties of a ship's hull which are varied in twenty discrete steps along the length.

A stepping relay control panel circuit designed to control three such stepping relays plus the imposing and removing of the initial conditions is shown in Fig. 7. The following description of this circuit is taken directly from Ref. 6.

Relay F is the master pulsing relay, its pulsing rate depending upon the two cams on the synchronous contactor and on the position of a remote switch.

Relay G, through normally closed contacts, passes pulses from relay F to the coil of stepping relay A. When stepping relay A reaches position 40, relay G is energized and no longer passes pulses. Stepping relay A then stops. Relays H and J perform the same functions for stepping relays B and C.

These three relays G, H and J also play an important part in imposing the initial conditions. When all three of these relays are energized (when all three stepping relays are on contact 40) power is furnished to the coil of relay L, which is then closed. This removes power from the "locking" contacts on relay M.

Relays L, M and N perform the functions of automatically imposing and removing the initial conditions. The initial conditions are imposed as soon as all three stepping relays reach contact 40.

When relay N is closed, the initial-condition relays are energized, thereby removing the initial conditions. Relay N is controlled by normally-open contacts on relay M. If relay M is momentarily energized it remains closed by virtue of its "electrically locking" contacts. These contacts obtain power from normally-closed contacts on relay L. (As long as relay M is closed, relay N is closed and all initial conditions are removed.) If relay L is energized (all stepping relays on contact 40) relay M "drops out" and the initial conditions are restored. The initial conditions are not removed until relay M is again energized which is done as soon as any one of the stepping relays reaches contact 1.

The stepping relays always stop on contact 40. When they are in this position relays G, H and J are energized and no longer furnish driving pulses to their respective stepping relays. Relay L is energized, removing power from the "locking" contacts of relay M. Relays M and N are inoperative, no power is furnished to the initial-condition relays and the initial conditions are imposed.

Relay O is the starting relay, controlled by the remote-control momentary-contact starting button, S_6. When this switch is closed momentarily, relay O is energized as soon as the next pulse is furnished by the microswitch on the one-per-second cam. Relay O will then remain closed until relay N closes. As soon as contact 40 is left, relays F, G and H open and pulses are continued to be supplied to the stepping relays. Simultaneously relay L drops out, energizing the "locking" contacts of relay M. At the instant any one on the stepping relays reaches contact 1 relay N closes and remains closed. This immediately removes the initial conditions and deenergizes the starting relay O.

In case any one or more of the three stepping relays are not used the corresponding switches S_A, S_B or S_C should be closed. This will then permit normal operation of relay L.

Fig. 8 is a photograph of a complete computer set up to solve a fourth order differential equation with variable coefficients. Fig. 9 is a box-type diagram of the same set-up identifying the various components of the system. Diagrams showing the external circuits of the amplifier combinations for performing the particular operations required are given in the following discussion of the individual problems.

DISCUSSION AND RESULTS

Preliminary to the primary object of this investigation, there is presented the solutions to two simple beam problems as determined by the analog computer. Such determinations were made as part of a program of familiarization with the technique to be used and to develop a facility in setting up and operating the components of the electrical circuits which become integral parts of the computer network used in the solution of the main problem.

Part I

The first preliminary problem was the determination of the static deflection under uniform load of a horizontally supported beam of constant cross section which is small in comparison with its length. Two different types of end fixity are demonstrated; viz., clamped and hinged.

The differential equation of the elastic curve of such a beam is given as[9]

$$EI \frac{d^4 y}{dx^4} = w(x) . \qquad (1)$$

Where: w(x) is the load per unit length along the beam.

y = vertical deflection of the beam at any point x.

x = distance along the beam measured from one end.

E = Young's Modulus of Elasticity.

I = area moment of inertia of a cross section of the beam with respect to the centroidal axis.

Bending moment and shear force at any point x along the beam are

Bending Moment: $M_{(x)} = EI \frac{d^2y}{dx^2}$.

Shear: $Q_{(x)} = EI \frac{d^3y}{dx^3}$.

Type I Beam Clamped at Both Ends

End Conditions: Zero slope and zero deflection at each end. These boundary conditions are expressed as

$y(0) = y'(0) = y(L) = y'(L) = 0.$

Theoretical Solution:[9]

$$y(x) = \frac{W(x)}{12\ EI}\left(\frac{L^2}{2}x^2 - Lx^3 + \frac{x^4}{2}\right)$$

$$y(max) = \frac{W(x)L^4}{384\ EI} \qquad @\ x = L/2.$$

The computer equation is set up by making a change of the independent variable in the original equation (1). The independent variable x is changed to t, time in seconds, and the length of the original beam is expressed at T, total elapsed time for solution in seconds.

Thus $x = \frac{L}{T}t$, and $\frac{d^n(\ )}{dx^n} = \frac{L^n}{T^n}\frac{d^n(\ )}{dt^n}$

the computer equation is then

$$EI\frac{d^4y}{dt^4} = \frac{L^4}{T^4}W(x) \tag{2}$$

The computer circuit for the solution of this equation is given in Fig. 10. The end conditions $y(0) = y'(0) = 0$ are obtained by initially shorting the feedback capacitors of A_3 and A_4. Shear force and bending moment have definite values at the ends of the beam and are simulated by battery voltages -Va and Vb respectively initially applied to the capacitors of A_1 and A_2. As the values of shear force and bending moment are unknowns, so the applied voltages -Va and Vb to be applied are at first unknown.

Vb was made fixed at about six volts and -Va was varied until a correct solution was obtained. A constant battery voltage V was applied as input voltage to A_1 to simulate the uniform loading, w(x), of the beam, and was measured in terms of recorder deflection units. The outputs of A_4 and A_5 were connected through amplifiers to channels 1 and 2 of the Brush recorder for recording oscillographs of -y' and y.

A correct solution was obtained in the following manner: With a constant input voltage V of about 1.3 volts applied to A_1, Vb constant at 6 volts, and -Va set at an arbitrary value, all initial conditions were imposed by closing the initial condition relay switches. The problem was started by de-energizing all the initial condition relays simultaneously thus releasing the end conditions. Several trials were made with different settings of the potentiometer controlling the battery voltage -Va before the proper end conditions were satisfied and recorded on the oscillograph. Fig. 11 shows the solution of the problem as recorded. The length of the solution, T, on the oscillograph is T = 3.6 seconds. V = 1.3.

Then $\frac{w(x)}{EI} L^4 = VT^4 = 1.3 \times 167.96 = 218.2$

$$y(max) \text{ theoretical} = \frac{218.2}{384} = 0.568.$$

$$y(max) \text{ from Fig. 11} = 0.550.$$

<u>Type II</u> Beam Hinged at Both Ends

End Conditions: Zero deflection and zero bending moment at each end. These boundary conditions are expressed as

$$y(0) = y''(0) = y(L) = y''(L) = 0.$$

Theoretical solution: [9]

$$y(max) = \frac{5}{384} \frac{w(x) L^4}{EI} \qquad @ x = L/2$$

The computer equation is the same as for Type I, but a change is necessary in the computer circuit due to the change in end conditions. The computer circuit for the solution of this problem is given in Fig. 12. The end conditions $y(0) = y''(0) = 0$ were obtained by shorting the feedback capacitors of A_2 and A_4. Shear force and slope have definite values at the ends of the beam and are simulated by battery voltages -Va and Vb respectively applied to capacitors of A_1 and A_3. The correct solution is obtained as before for Type I. Several trials again were necessary in varying the potentiometer of -Va to obtain the proper end conditions.

Fig. 13 shows the solution of the problem as recorded on the oscillograph. The length of the beam T, on the oscillograph is measured at T = 2.96 seconds. V = 1.3.

Then $\frac{w(x)\,L^4}{EI} = VT^4 = 1.3 \times 76.7 = 99.6$

$$y(max) \text{ theoretical} = \frac{5 \times 99.6}{384} = 1.30$$

$$y(max) \text{ from Fig. 13} = 1.15$$

The second preliminary problem was the determination of the first three normal modes of lateral vibration of a uniform free-free beam considering the effects of bending deflection only. The vibrating beam is considered loaded by inertia forces due to its own mass and acceleration.

The differential equation of motion of the elastic curve of such a beam is given by [10]

$$EI\,\frac{d^4y(x,t)}{dx^4} = \mu\,\frac{\partial^2 y(x,t)}{\partial t^2} \qquad (3)$$

where $\mu = \frac{A\gamma}{g}$, the mass distribution along the beam

γ = the density of the material of the beam.

A = the cross sectional area of the beam.

g = the acceleration due to gravity.

$\mu \frac{\partial^2 y(x,t)}{\partial t^2}$ = the inertia forces acting on the beam.

y = vertical deflection of the beam at x.

x = distance along the beam measured from one end.

E = Young's Modulus of Elasticity.

I = Area moment of inertia of a cross section of the beam with respect to the centroidal axis.

It is assumed that $y(x,t) = X(x)\, e^{j\omega t}$.

Where X(x) is a function only of x, and is independent of time, $e^{j\omega t}$ represents sinusoidal oscillations of frequency ω.

Then $$\frac{\partial^2}{\partial t^2} y(x,t) = -X(x)\omega^2 e^{j\omega t}$$

and equation (3) becomes

$$EI\frac{d^4X}{dx^4} - \mu\omega^2 X = 0$$

or

$$\frac{EI}{\mu\omega^2}\frac{d^4X}{dx^4} - X = 0 \qquad (4)$$

The computer equation is set up as before by making a change of the independent variable in the original equation (4). The independent variable, x, is changed to t, time in seconds, and the length of the beam is expressed as T, total elapsed time for solution in seconds.

Then $x = \frac{L}{T} t$ and $\frac{d^n(\ )}{dx^n} = \frac{L^n}{T^n} \frac{d^n(\ )}{dt^n}$

The computer equation becomes

$$\frac{EI\ T^4}{\mu\omega^2 L^4} \frac{d^4X}{dt^4} - X = 0 \qquad (5)$$

For simplicity we let

$$\alpha_n^2 = \frac{\mu\omega_n^2 L^4}{EI}$$

and $\omega_n = \alpha_n \sqrt{\frac{EI}{\mu L^4}}$ the natural frequency of vibration for the n^{th} mode. In addition, for the computer equation (5) we denote

$$C = \frac{T^4}{\alpha_n^2}$$

so that the computer equation reduces to

$$C \frac{d^4X}{dt^4} - X = 0 \qquad (6)$$

For the computer solution, the C was given a value of unity (1 megohm) and the problem was solved by finding a length, T, on the oscillograph solution for which the simulated end conditions as determined by the beam supports were met.

The computer circuit for the solution of equation (6) is given in Fig. 14. The end conditions to be satisfied for a free-free beam are that the bending moment and shear force at each end are zero. These boundary conditions are expressed as

$$X''(0) = X'''(0) = X''(L) = X'''(L) = 0$$

To satisfy these end conditions on the computer, the feedback capacitors of A_1 and A_2 are initially shorted. As there is a definite but unknown slope and deflection at each end of the beam, these are simulated on the computer by battery voltages -Va and Vb respectively initially applied to the capacitors of A_3 and A_4. As before, Vb was fixed at about six volts and -Va was varied for different trial solutions until the end conditions of zero shear force and bending moment were satisfied.

The outputs of A_1 and A_2 were connected through amplifiers to the two channels of the Brush recorder for recording oscillographs of X'' and -X'''. Correct solutions showing the fulfillment of the end conditions required were obtained when the minimum or maximum of X'', depending upon the number of the mode, passed through the zero axis. The -X''' curve was used in measuring the length, T, of the solution. This curve was used in preference to X'' because the -X''' curve has a definite finite slope at each end of the solution.

Correct solutions of the frequency for the first three normal modes of vibration of the beam were obtained in a manner similar to that previously described. Vb was made constant and -Va was initially set at an arbitrary value. All initial end conditions were imposed by closing the initial condition relay switches. The problem was started by simultaneously releasing all the end conditions. Several trial settings of the potentiometer controlling the voltage -Va were necessary before a correct solution for each mode was obtained.

Solutions of the first mode were quite readily obtained, but for the second and third modes the setting of the potentiometer controlling the voltage -Va was found to be very critical. The inherent slight instability of the amplifiers used and variations in power supply voltage were enough to cause trouble in repetition of solutions. Many trials were necessary to obtain a few correct solutions. Fig. 15 shows the correct solution oscillographs obtained for the first three modes. The results obtained for α_1, α_2, and α_3 checked very closely with those given in the Appendix of Den Hartog[10]:

	Mode	Den Hartog	Computer
α_1	1	22.4	22.4
α_2	2	61.7	61.78
α_3	3	121.0	121.0

In evaluating the results of the preliminary problems, several factors were noted as being most important in the obtaining of correct solutions on the analog computer of the laboratory type used. These factors are mentioned briefly here and were kept constantly in mind in setting up and operating the computer for the main problem.

1. The power supply to the individual amplifiers must be as required by the amplifiers and as non-variant as practically possible.

2. The tube heater voltage of about [illegible] volts is best obtained from a storage battery.

3. Each individual amplifier should be finely balanced for zero gain and this balance should be kept by frequent check.

4. The amplifiers should be finely balanced for zero gain in the computer network in groups of 2, 3, and 4, and be kept in balance by frequent check.

5. All precautions should be taken that the 60 cycle "pick-up" by the network be kept to a minimum that can be tolerated by the system.

6. As the measurement of length of time as recorded on the oscillograph is most critical in arriving at a solution, pen lag of the recorder should be kept at a minimum, and the 110 volt A.C. 60 cycle

driving the synchronous motor of the Brush recorder should be carefully regulated. Pen lag effect can be effectively reduced by increasing the amplifier output voltage recorded on the oscillograph.

7. The potentiometer controlling the battery voltage -Va had three degrees of fineness of control which was found quite necessary for solutions of the higher modes. Va for the second mode was found to be very close to the Va for the third mode.

It was felt that the accuracy of the results obtained in the preliminary work was well within the accuracy of the laboratory type equipment used. As the technique on the part of the operators of the equipment improved both in setting up the problem and in checking the balance of the amplifiers in the network, the results obtained by the computer definitely improved in accuracy.

Part II

For the purpose of this study, the vibrating ship is replaced by an ideal floating beam with uniform elastic and inertia properties. The differential equation for the vertical motion of the elastic curve of this floating beam has been derived[11] including the effects of bending and shear deflection, rotary inertia, external loading, damping force, and buoyancy force. The problem here considered is the solution of the differential equation for the vertical motion of the elastic curve of the vibrating ship considering first bending deflection only, and second, bending and shear deflection with rotary inertia effect.

The complete differential equation of the elastic curve is given by[11]

$$\frac{\partial^2}{\partial x^2}\left[EI_A\frac{\partial^2 y(x,t)}{\partial x^2} - I_\mu\frac{\partial^2 y(x,t)}{\partial t^2}\right] + \left[1 + \frac{\partial^2}{\partial t^2}\frac{I_\mu}{KAG} - \frac{\partial^2}{\partial x^2}\frac{EI_A}{KAG}\right]\left[\mu\frac{\partial^2 y(x,t)}{\partial t^2}\right] = 0 \qquad (7)$$

where each term has a physical interpretation.

The first term

$$\left[EI_A\frac{\partial^2 y(x,t)}{\partial x^2} - I_\mu\frac{\partial^2 y(x,t)}{\partial t^2}\right]$$

is the sum of the moments due to elastic deformation in bending and rotary inertia. The differential operator $\frac{\partial^2}{\partial x^2}$ reduces this to an equivalent distributed load.

The term $\mu\frac{\partial^2 y(x,t)}{\partial y^2}$ is the distributed load due to translatory inertia.

Considering first the effect of bending deflection only, the differential equation reduces to

$$\frac{\partial^2}{\partial x^2}\left[EI_A \frac{\partial^2 y(x,t)}{\partial x^2}\right] + \mu \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \qquad (8)$$

where μ = the mass distribution along the beam and includes the virtual mass which is an equivalent mass added to that of the ship to represent the inertia effect of the water accelerated with the ship's vibration.

For sinusoidal oscillations of frequency ω, it is assumed that $y(x,t) = X(x)\, e^{j\omega t}$

where $X(x)$ is a function only of x distance along the beam and is independent of time.

$e^{j\omega t}$ represents sinusoidal oscillations of frequency ω.

Then $\frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 e^{j\omega t} X(x)$.

and assuming that E is a constant and that I and μ are functions only of x along the beam, equation (8) becomes

$$\frac{d^2}{dx^2}\left[EI \frac{d^2X}{dx^2}\right] - \mu\omega^2 X = 0. \qquad (9)$$

We let $I = I_o i(x)$ and $\mu = \mu_o \beta(x)$ where I_o and μ_o are maximum values of moment of inertia and mass respectively.

To set up the computer equation, a change in the independent variable is again necessary. Where x was the independent variable, let t, time in seconds on the computer solution be the new variable. Then

$$t = \frac{T}{L} x$$

where L is the length of the ship actually, T is the length of the ship on the computer solution. Where $0 \leq x \leq L$ before, now $0 \leq t \leq T$ for the computer.

In general, $$\frac{d^n(\ )}{dx^n} = \frac{L^n}{T^n} \frac{d^n(\ )}{dt^n}$$

and equation (9) becomes

$$\frac{T^4}{L^4} \frac{d^2}{dt^2}\left(EI \frac{d^2X}{dt^2}\right) - \mu\omega^2 X = 0. \qquad (10)$$

Substituting for I and μ, E = a constant, and dividing through by ω^2, and μ_o, we have

$$\frac{T^4 \, EI_o}{\mu_o L^4 \omega^2} \frac{d^2}{dt^2}\left[i(t) \frac{d^2X}{dt^2}\right] - \beta(t)X = 0 \qquad (11)$$

Letting $\alpha_n^2 = \dfrac{\mu_o \omega_n^2 L^4}{EI_o}$ and $C = \dfrac{T^4}{\alpha_n^2}$

the natural frequency of vibration of the n^{th} mode is

$$\omega_n = \alpha_n \sqrt{\frac{EI_o}{\mu_o L^4}}$$

The computer equation becomes

$$C \frac{d^2}{dt^2}\left[i(t) \frac{d^2X}{dt^2}\right] - \beta(t) X = 0 \qquad (12)$$

In this computer equation, the bending

moment is proportional to $i(t)\,\frac{d^2X}{dt^2}$ and the shear is proportional to $\frac{d}{dt}\left[i(t)\,\frac{d^2X}{dt^2}\right]$

As the ship acts as a free-free beam, the end boundary conditions on the problem are that the bending moment and shear are zero at each end. The boundary conditions are expressed as

$$i(o)\,\frac{d^2X}{dt^2} = \frac{d}{dt}\left[i(o)\frac{d^2X}{dt^2}\right] = i(T)\,\frac{d^2X}{dt^2} = \frac{d}{dt}\left[i(T)\frac{d^2X}{dt^2}\right] = 0$$

The computer circuit for the solution of equation (12) is given in Fig. 16. The end conditions of zero bending moment and shear are satisfied on the computer by initially shorting the feedback capacitors of A_3 and A_2 respectively. As the slope and deflection at each end of the beam are unknown, these are simulated on the computer by battery voltages -Va and Vb respectively applied initially to the capacitors of A_5 and A_6. Vb was fixed at about 6 volts and -Va was varied for the different trial solutions for the various modes until the end conditions of zero shear and bending moment were satisfied.

On Table I is tabulated the original data on the APA 87, the ship for which frequency of vibration was desired. This ship has the following general characteristics

L	= Load water line length	= 400'
B	= Molded breadth	= 58'
D	= Molded depth	= 37'
d	= Full load draft	= 15' 6"
Δ	= Full load displacement	= 6800 tons
E	= Young's Modulus of elasticity	= 1.93×10^6 tons/ft^2
I_o	= Maximum area moment of inertia	= 2628 ft^4
μ_o	= Maximum total mass per unit L	= 2.006 $\frac{\text{tons sec}^2}{\text{ft}^2}$

For the purposes of calculation, the ship was divided into 20 parts of 20 foot lengths each. For each section there is tabulated the moment of inertia and mass. Fig. 17 shows the distribution of mass and moment of inertia of the APA 87 as introduced into the computer. On Table II there is tabulated $i(t)$ and $\beta(t)$ which were simulated by means of 40 steps, 4 steps per second, on the stepping relay and resistor panel. The resistances in megohms added for each two steps are tabulated on Table II. The negative sign before the resistance tabulated indicates that this resistance was subtracted to give the proper value of $i(t)$ or $\beta(t)$ for that step. For $i(t)$, the resistances listed are introduced as input resistance to A_4 and for $\beta(t)$, the resistors listed are introduced as feedback resistance to A_1 to satisfy equation (12).

As the stepping relay resistor panel, Fig. 6, was originally planned, the length of solution on the computer should have been T = 10 seconds. However, as the stepping relays used stopped the problem immediately upon reaching step 40, the length of the solution, T = 9.75 seconds. This condition could have been corrected by rewiring the stepping relays' control relay circuit to provide for a quarter second pause on step 40 before the initial condition relays again imposed the end condition and stopped the problem. The authors of this paper did not feel at liberty to change the equipment in this manner, and felt that the quarter second lost could be accounted for in the solution knowing that T was actually 9.75 seconds. Even though the quarter second in the length of the computer solution meant ten feet in the length of the ship, this "lost" section of the bow does not materially affect the vibratory characteristics of the vessel.

Fig. 9 shows schematically the arrangement of the computer and complete network of controls and power supply. Fig. 8 is a photograph of the complete network. The outputs of A_2 and A_3 are shown connected through amplifiers to the two channels of the Brush recorder for recording oscillographs of $\frac{d}{dt}\left[i(t)\,\frac{d^2X}{dt^2}\right]$ and $-i(t)\,\frac{d^2X}{dt^2}$.

As in the solution of the uniform free-free beam, correct solutions showing the fulfillment of the end conditions of zero shear and bending moment were obtained when the minimum or maximum of y(t) or, depending upon the number of the mode, passed through the time axis. The $\frac{d}{dt}\left[I(t)\,\frac{d^2y}{dt^2} \right]$ curve was used in measuring the length of the solution, T.

With the components of the computer connected [illegible] as described in the Equipment and Procedure section of this paper, correct solutions were obtained by first varying -Va from trial to trial until end conditions were met. Then C was varied from trial to trial until a solution of length T = [illegible] seconds was reached. It was found that with each trial value of C introduced, -Va had to be changed slightly for the end conditions to be satisfied for that trial.

The oscillographs of solutions obtained for the first two modes are given in Fig. 18 and Fig. 19. It was impossible due to the limitations of the equipment used to obtain a correct solution satisfying the end conditions for the third mode. Many attempts were made and the near solutions obtained were plotted and [illegible] plotted on Fig. 20. From this figure, an approximate value of C necessary for a correct solution could be obtained. This method of plotting [illegible]

This correct solution was developed by [illegible] in Ref. (6) and was extensively used by him when correct oscillograph solutions were difficult or impossible to obtain.

Theoretical solutions of the frequency of normal modes of vibration of the APA 87 have been calculated by graphical methods and by a calculator designed by IBM[8]. Comparative results are listed below.

Frequency of Vibration - Normal Modes

Bending Only

Mode	Computer		IBM
	C	rad/sec.	rad/sec.
1	6.0	12.4	11.5
2	0.98	29.6	28.4
3	0.25	60.6	

Considering now the effect of shear deflection and rotary inertia in addition to the bending deflection effect just discussed:

Equation (7) was the complete differential equation of the elastic curve in which the term

$$- I_{\mu} \frac{\partial^2 y(x,t)}{\partial t^2}$$

had the physical interpretation as a moment due to elastic deformation in rotary inertia. This term was reduced by the differential operator $\frac{\partial^2}{\partial x^2}$

to an equivalent distributed load. I_μ is the mass moment of inertia of a cross section area per unit length of the beam.

In equation (7) is also included the term $\frac{EI_\mu}{KAG}$ which is also reduced by the operator $\frac{\partial^2}{\partial x^2}$ to an equivalent distributed load. This is the only term left in the original equation (7) representing the effect of shear deflection as the term $\frac{\partial^2}{\partial t^2}\frac{I_\mu}{KAG}\,\mu\,\frac{\partial^2 y(x,t)}{\partial t^2}$ has been left out in setting up the computer equation.

Regrouping the terms of equation (7), the following fourth order differential equation is obtained which considers the effect of bending deflection and an approximation to the effects of shear deflection and rotary inertia:

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 y(x,t)}{\partial x^2} - I\mu\left(\frac{E}{KAG} + \frac{1}{A}\right)\frac{\partial^2 y(x,t)}{\partial t^2}\right] + \mu\frac{\partial^2 y(x,t)}{\partial t^2} = 0. \quad (13)$$

where I is area moment of inertia of a cross section area.

In addition to the assumption that $y(x,t) = X(x)\, e^{j\omega t}$, it is assumed that over the length of the ship the term $\left(\frac{E}{KAG} + \frac{1}{A}\right)$ as a constant. [illegible]

of the tabulated data for the ship considered, Table VII, shows the reasonableness of this assumption. Denote this constant as B, then

$$B = \left(\frac{E}{KAG} + \frac{1}{A}\right).$$

With I and μ as functions only of x distance along the beam and independent of time, we have

$$\frac{d^2}{dx^2}\left[EI\frac{d^2X}{dx^2} + I\mu B\omega^2 X\right] - \mu\omega^2 X = 0.$$

(14)

As was done for bending deflection only, let $I = I_0 i(x)$ and $\mu = \mu_0 \beta(x)$ and set up the computer equation by making a change in the independent variable, x, in the original equation (14) to t, time in seconds on the computer solution. Then $t = \frac{T}{L}x$.

Making the necessary substitutions and dividing through by ω^2 and μ_0, equation (14) becomes

$$\frac{T^4 EI_0}{\omega^2 L^4 \mu_0}\frac{d^2}{dt^2}\left[i(t)\frac{d^2X}{dt^2}\right] + \frac{I_0 BT^2}{L^2}\frac{d^2}{dt^2}\left[i(t)\beta(t)\right]X - \beta(t)X = 0$$

(15)

Letting $a_n = \omega_n\sqrt{\frac{\mu_0 L^4}{EI_0}}$; $C = \frac{T^4}{a_n^2}$

and H = $\frac{I_o \beta T^2}{L^2}$, equation (15) becomes

$$C \frac{d^2}{dt^2}\left[i(t) \frac{d^2X}{dt^2}\right] + H \frac{d^2}{dt^2}\left[i(t)\beta(t)\right] x - \beta(t)X = 0$$

The natural frequency of the vibration in the n^{th} mode is

$$\omega_n = a_n \sqrt{\frac{EI_o}{\mu_o L^4}}$$

In order to obviate the necessity of setting up a new computer circuit to introduce the product term of the two variables, i(t) and β(t), it was felt a good approximation could be attained by selecting from the data available a representative average value of the product term $\left[i(t)\ \beta(t)\right]$ which could be assumed constant over the length of the beam. This was done and with the regrouping of constants in the second term of equation (16) to a single constant, D,

$$D = H \left[i(t)\ \beta(t)\right]$$

and the computer equation becomes

$$C \frac{d^2}{dt^2}\left[i(t) \frac{d^2X}{dt^2}\right] + D \frac{d^2X}{dt^2} - \beta(t)X = 0. \qquad (17)$$

The end conditions for a free-free beam are that bending moment and shear are zero at both ends. In the previous problems discussed, these were proportional to the second and third derivatives of the deflection respectively. When in addition to bending deflection, shear and rotary inertia effects are considered, the bending moment and shear force are given as [12],

$$M = EI\left(\frac{\partial^2 y}{\partial x^2} + \frac{\mu\omega^2}{KAG}\, y\right)$$

$$Q = \frac{EI}{1 - \frac{\mu I \omega^2}{KAG}}\left[\frac{\partial^3 y}{\partial x^3} + \frac{\mu\omega^2}{KAG} + \frac{\mu\omega^2}{E}\,\frac{\partial y}{\partial x}\right].$$

However, in setting up the computer equation the assumption was made that $\frac{\partial^2}{\partial t^2}\frac{I\mu}{KAG}\mu\frac{\partial^2 y}{\partial t^2}$ could be neglected for a good approximation of the effect of shear and rotary inertia. Following this assumption, the bending moment is proportional to $i(t)\,\frac{d^2X}{t^2}$ and the shear is proportional to

$\frac{d}{it}\left[i(t)\,\frac{d^2y}{dt^2}\right]$ in the computer equation (17).

The boundary conditions for the solution of the computer equation are then expressed as

$$i(o)\,\frac{d^2X}{dt^2} = \frac{d}{dt}\left[i(o)\,\frac{d^2X}{dt^2}\right] = i(T)\frac{d^2X}{dt^2} = \frac{d}{dt}\left[i(T)\frac{d^2X}{dt^2}\right] = 0$$

The computer circuit for the solution of equation (17) is given in Fig. 16, where the dotted line with resistance C/D from the output of A_4 to the input of A_2 is included to accomplish the $D\,\frac{d^2X}{dt^2}$ term in the computer equation. The procedure for the solution of equation (17) is similar to that described for the solution of equation (12) where bending deflections only were considered. In addition to varying C, it is now necessary to vary the ratio C/D. As D remains constant for all modes, the ratio 1/D could have been introduced into the computer circuit instead of C/D and .5/C would have been the variable introduced as a feedback resistor on A_4, necessary from trial to trial for each mode.

The oscillographs of solutions obtained for the first four modes are given in Figs. 21 through 24 inclusive.

It was found that the addition of the C/D input resistance to A_2 from A_4 gave considerable stability to the computer circuit in the range of resistance used. The alternative of using a constant 1/D ratio as mentioned above as an input resistance was not as successful in stabilizing the circuit and so was not used.

Theoretical solutions of the frequency of normal modes of vibration of the APA 87 have been determined by a calculator designed by IBM for bending and shear deflection and rotary inertia effects.[8] Comparative results are listed below.

Frequency of Vibration - Normal Modes

Bending and Shear Deflection and Rotary Inertia

Mode		Computer	IBM
	C	rad/sec.	rad/sec.
1	7.00	11.29	10.30
2	1.685	23.02	19.95
3	0.725	35.00	30.02
4	0.410	46.60	39.28

The frequencies obtained by means of the analog computer when the effects of bending deflections only are considered are approximately five to eight percent higher than the IBM solutions. The analog computer results are calculated directly from the oscillographs and have not been corrected for recorder pen lag or power supply frequency variation. The latter is an important factor, correction for which should be made for better accuracy of results. Line frequency variation has an effect both on the measurement of "T" on the recorder tape and on the balance of the operational amplifiers necessary for repetition of solutions.

Another source of error in the computer effecting the accuracy of the results lies in the stepping relays and resistor panels used. Every effort was made to have accurate resistances on each step for the simulation of mass and moment and inertia. However, the many plug-in connections of resistors in stacks on the resistor panels introduced inaccuracies in the actual resistance obtained for each step. Then too, it is known that the bridging contacts of the stepping relays did not always perfectly bridge from one step to the next.

It is felt that the results were well within the accuracy of the computer network itself, and that corrections made to the oscillograph records as mentioned above would improve the precision of the values of frequencies of vibration obtained.

The results obtained when, in addition to bending deflection effect, an approximation to the effects of shear deflection and rotary inertia was considered, are progressively higher, (from about ten percent for the first mode to about eighteen percent for the fourth mode) than those obtained by the IBM computer.

It is apparent that these deviations which increase with the higher modes result from something more than the inaccuracies in the computer network. The assumptions made to approximate the effects of shear deflection and rotary inertia in setting up the computer equation must, in a large part, account for the increasing error. These assumptions were necessary to avoid complicating the present computer network to a degree out of proportion to the actual effect of shear and rotary inertia on the frequencies of vibration of the ship.

With a computer constructed of more precise and stable components, the product term $\left[i(t)\beta(t)\right]$ could be introduced as a variable. A network could be set up to solve a computer equation which includes the term

$$\frac{\partial^2}{\partial t^2}\frac{I_s}{KAG}\mu\frac{\partial^2 y(x,t)}{\partial t^2}.$$

The precision of the values of frequencies obtained from such an analog computer network should be definitely better than in the present instance.

CONCLUSIONS

Solutions to many engineering problems of practical interest involving higher order differential equations with variable coefficients may be obtained by means of a relatively simple and inexpensive electronic analog computer. Solutions so obtained are well within the accuracy necessary for most engineering purposes.

The accuracy of solutions obtained are limited by the precision of the computer components used and regulation of the associated power supplies. The assumptions made in reducing an exact differential equation to a computer equation are in a large part necessitated by the precision of the apparatus used. Low precision of components, for instance, would limit the number of amplifiers and variables introduced into the computer network for a given desired accuracy. As the complexity of the network increases, so must the precision and stability of the components increase.

Instead of introducing another variable of the original equation into the computer with the consequent additional amplifiers and component circuits, a constant average effect of the variable may be introduced. Errors resulting from such assumption of average effect must be weighed against those resulting from lack of precision of the circuit elements. In the case of the APA 67 in this paper, the constant average effect of shear deflection and rotary inertia embodied in the term $\left[i(t)\beta(t) \right]$ would appear not to be representative of the true effect. However, if the term $\frac{\partial^2}{\partial t^2} \frac{I_\mu}{KAG} \mu \frac{\partial^2 y(x,t)}{\partial t^2}$ had been retained in setting up the computer equation, there would undoubtedly have been closer agreement in results obtained. In the latter case, the average effect was assumed to be zero, an obviously over simplification in light of the results obtained by the computer.

Economy of time and personnel is the primary benefit of such an analog computer as described in this paper. With the necessary equipment available and having familiarity with the operating procedures, the solution of higher order differential equations with variable coefficients would be a matter of a few hours for a single operator.

The analog computer is especially adaptable to the solution of design problems where the study of the effects of varying design parameters may be conducted with little effort by simple external changes to a single computer network.

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C A L C U L A T I O N S

CALCULATION OF FREQUENCY OF VIBRATION

Bending Only

Constants used:

$T = 9.75$ sec. $T^2 = 95.0625$ sec^2

$$\sqrt{\frac{EI_o}{\mu_o L^4}} = \sqrt{\frac{1.93\times10^6 \times 2628}{2.006 \times 256 \times 10^8}} = 0.314$$

1st Mode

$C = 6.0$

$$\alpha_1 = \frac{T^2}{\sqrt{C}} = \frac{95.0625}{\sqrt{6.0}} = 38.85$$

$$\omega_1 = \alpha_1\sqrt{\frac{EI_o}{\mu_o L^4}} = 38.85 \times 0.314 \text{ rad./sec.}$$

$\omega_1 = 12.4$ rad/sec.

2nd Mode

$C = 0.98$

$$\alpha_2 = \frac{T^2}{\sqrt{C}} = \frac{95.0625}{\sqrt{0.98}} = 95.1$$

$$\omega_2 = \alpha_2\sqrt{\frac{EI_o}{\mu_o L^4}} = 95.1 \times 0.314 \text{ rad./sec.}$$

$\omega_2 = 29.8$ rad./sec.

3rd Mode

$C = 0.25$

$$\alpha_3 = \frac{T^2}{\sqrt{C}} = \frac{95.0625}{\sqrt{0.25}} = 190.125$$

$$\omega_3 = \alpha_3\sqrt{\frac{EI_o}{\mu_o L^4}} = 190.125 \times 0.314 \text{ rad./sec.}$$

$\omega_3 = 60.6$ rad./sec.

CALCULATION OF FREQUENCY OF VIBRATION

Bending, Shear, and Rotary Inertia

Constants used:

$T = 9.75$ sec $T^2 = 95.0625$ sec.^{2}

$\frac{EI_o}{\mu_o L^4} = 0.0986$

From Table III $B = 1.304\ ^1/ft^2$; $\left[i(t)\ \beta(t)\right] = 0.533$

$H = \frac{I_o B\ T^2}{L^2} = \frac{2628 \times 1.304 \times 95.0625}{160,000}$

$H = 2.039$ sec.^{2}

$D = H\left[i\ (t)\ B\ (t)\right] = 2.039 \times .533$

$D = 1.087$ sec.^{2}

1st Mode

$C = 7.0$

$\alpha_1^2 = \frac{T^4}{C} = \frac{9040}{7} = 1291$

$\omega_1^2 = \alpha_1^2\ \frac{EI_o}{\mu_o L^4} = 1291 \times 0.0986 = 127.5$

$\omega_1 = 11.29$ rad./sec.

2nd Mode

$C = 1.685$

$\alpha_2^2 = \frac{T^4}{C} = \frac{9040}{1.685} = 5360$

$\omega_2^2 = \alpha_2^2\ \frac{EI_o}{\mu_o L^4} = 5360 \times 0.0986 = 530$

$\omega_2 = 23.02$ rad./sec.

<u>3rd Mode</u>

$C = 0.725$

$$\alpha_3^2 = \frac{T^4}{C} = \frac{9040}{0.725} = 12{,}470.$$

$$\omega_3^2 = 12470 \times 0.0986 = 123[illegible]$$

$\omega_3 = 35.0$ rad./sec.

<u>4th Mode</u>

$C = 0.410$

$$\alpha_4^2 = \frac{T^4}{C} = \frac{9040}{0.41} = 22{,}050$$

$$\omega_4^2 = \alpha_4^2 \frac{EI_0}{\mu_0 L^4} = 22050 \times 0.0986 = 2175$$

$\omega_4 = 46.6$ rad./sec.

TABLE I [illegible]

DATA FOR CALCULATION OF NORMAL MODES OF VERTICAL VIBRATION

$L = 400$ ft. $E = 1.93 \times 10^6$ tons/ft.^{2}

Section Stern To Bow	I ft.^{4}	$\mu \frac{T sec^2}{Ft^2}$	K	A Ft.^{2}	KAG tons
0-1	617	0.1775	0.310	5.35	1.278
1-2	1157	0.6271	0.271	7.15	1.494
2-3	1586	0.9025	0.220	9.30	1.578
3-4	1895	1.1300	0.171	12.29	1.621
4-5	2146	1.2946	0.150	14.03	1.622
5-6	2334	1.3548	0.14[illegible]	13.75	1.579
6-7	2454	1.4568	0.15[illegible]	13.61	1.637
7-8	2532	1.9493	0.161	13.82	1.715
8-9	2585	1.9540	0.1[illegible]	14.44	1.737
9-10	2609	2.0060	0.14[illegible]	14.93	1.658
10-11	2623	1.9401	0.137	14.65	1.548
11-12	2628	1.9159	0.134	14.93	1.542
12-13	2628	1.6765	0.141	15.97	1.736
13-14	2614	1.4242	0.152	15.49	1.815
14-15	2599	1.2439	0.168	12.99	1.682
15-16	2493	1.0189	0.186	12.50	1.792
16-17	2281	0.7112	0.205	11.20	1.[illegible]66
17-18	1934	0.4483	0.225	10.14	1.759
18-19	1447	0.2548	0.245	7.99	1.509
19-20	738	0.2613	0.264	5.49	1.[illegible]17

TABLE II APA 87

i(t) and β(t) in Megohms as Introduced into Computer

I_o = 2628 ft. $I = I_o\, i(t)$

μ_o = 2 0060 tons sec.²/ ft.² $\mu = \mu_0 \beta(t)$

Stern to Bow Section	i(t)	ΔR MEG Δi(t)	ΔR MEG Breakdown	β(t)	ΔR MEG Δβ(t)	ΔR MEG Breakdown
0-1	0.235	0.235	.235	0.086	0.086	.003 .036
1-2	0.439	0.204	.046 .158	0.313	0.227	.041 .097 .089
2-3	0.604	0.165	.133 .032	0.450	0.137	.042 .095
3-4	0.720	0.116	.116	0.563	0.113	.058 .055
4-5	0.815	0.095	.045 .050	0.646	0.083	.058 .025
5-6	0.888	0.073	.053 .020	0.676	0.030	.030
6-7	0.934	0.046	.046	0.726	0.050	.035 .015
7-8	0.963	0.029	.014 .015	0.972	0.246	.119 .110 .013 .004
8-9	0.984	0.021	.021	0.975	0.003	.003
9-10	0.993	0.009	.005 .004	1.000	0.025	.025
10-11	0.998	0.005	.001 .004	0.968	$\bar{0}$.032	
11-12	1.000	0.002	.002	0.955	$\bar{0}$.013	
12-13	1.000	0		0.836	$\bar{0}$.119	
13-14	0.994	$\bar{0}$.006		0.711	$\bar{0}$.125	
14-15	0.989	$\bar{0}$.005		0.621	$\bar{0}$.090	
15-16	0.948	$\bar{0}$.041		0.508	$\bar{0}$.115	
16-17	0.868	$\bar{0}$.080		0.355	$\bar{0}$.153	
17-18	0.765	$\bar{0}$.103		0.224	$\bar{0}$.131	
18-19	0.572	$\bar{0}$.193		0.127	$\bar{0}$.097	
19-20	0.281	$\bar{0}$.291		0.130	+0.003	
ΔX=20′						

DATA FOR CALCULATION OF CONSTANTS B, H and D.

$$B = \left(\frac{E}{KAG} + \frac{1}{A} \right), \quad H = \frac{I_o B T^2}{L^2}, \quad D = H \left[i(t) / \beta (t) \right]$$

Section	E/KAG $1/FT^2$	$\frac{1}{A}$ $1/FT^2$	$\frac{E}{KAG} + \frac{1}{A}$ $1/FT^2$		$i(t)/\beta(t)$
0-1	1.511	0.187	1.698		0.022
1-2	1.291	0.140	1.431		0.137
2-3	1.222	0.107	1.329		0.272
3-4	1.190	0.081	1.271		0.405
4-5	1.188	0.071	1.259		0.526
5-6	1.222	0.073	1.295		0.600
6-7	1.180	0.073	1.253		0.677
7-8	1.124	0.072	1.196		0.935
8-9	1.111	0.069	1.180		0.959
9-10	1.163	0.067	1.230		0.993
10-11	1.247	0.068	1.315		0.965
11-12	1.250	0.067	1.317		0.955
12-13	1.111	0.062	1.173		0.836
13-14	1.062	0.065	1.127		0.706
14-15	1.147	0.077	1.224		0.615
15-16	1.076	0.080	1.156		0.481
16-17	1.033	0.085	1.118		0.308
17-18	1.097	0.098	1.195		0.171
18-19	1.279	0.125	1.404		0.073
19-20	1.730	0.182	1.912		0.036
			Σ = 26.083		Σ = 10.671
			$B_{AVE.}$ = 1.304		AVE. = 0.533

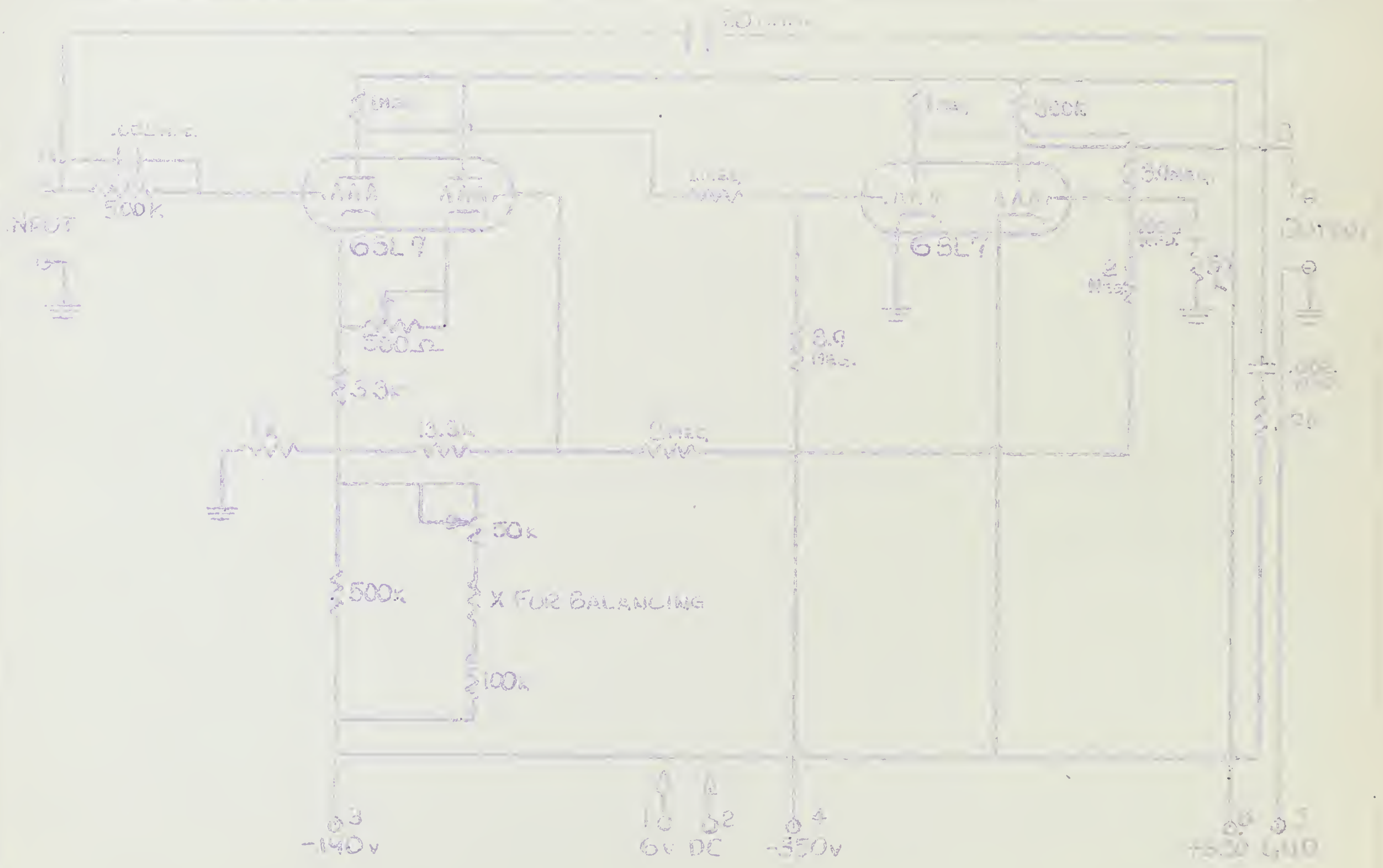


FIGURE 1 DC AMPLIFIER

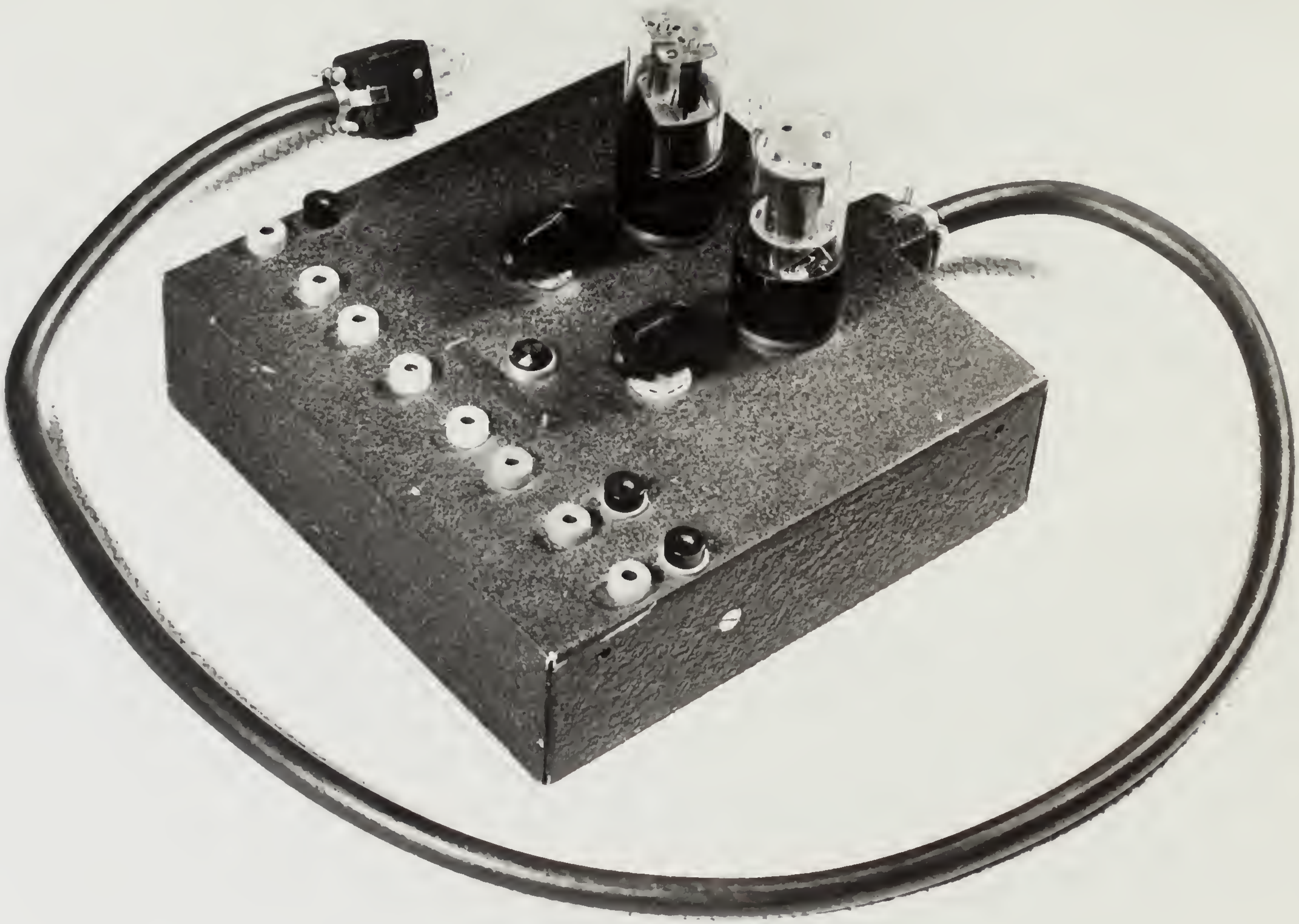
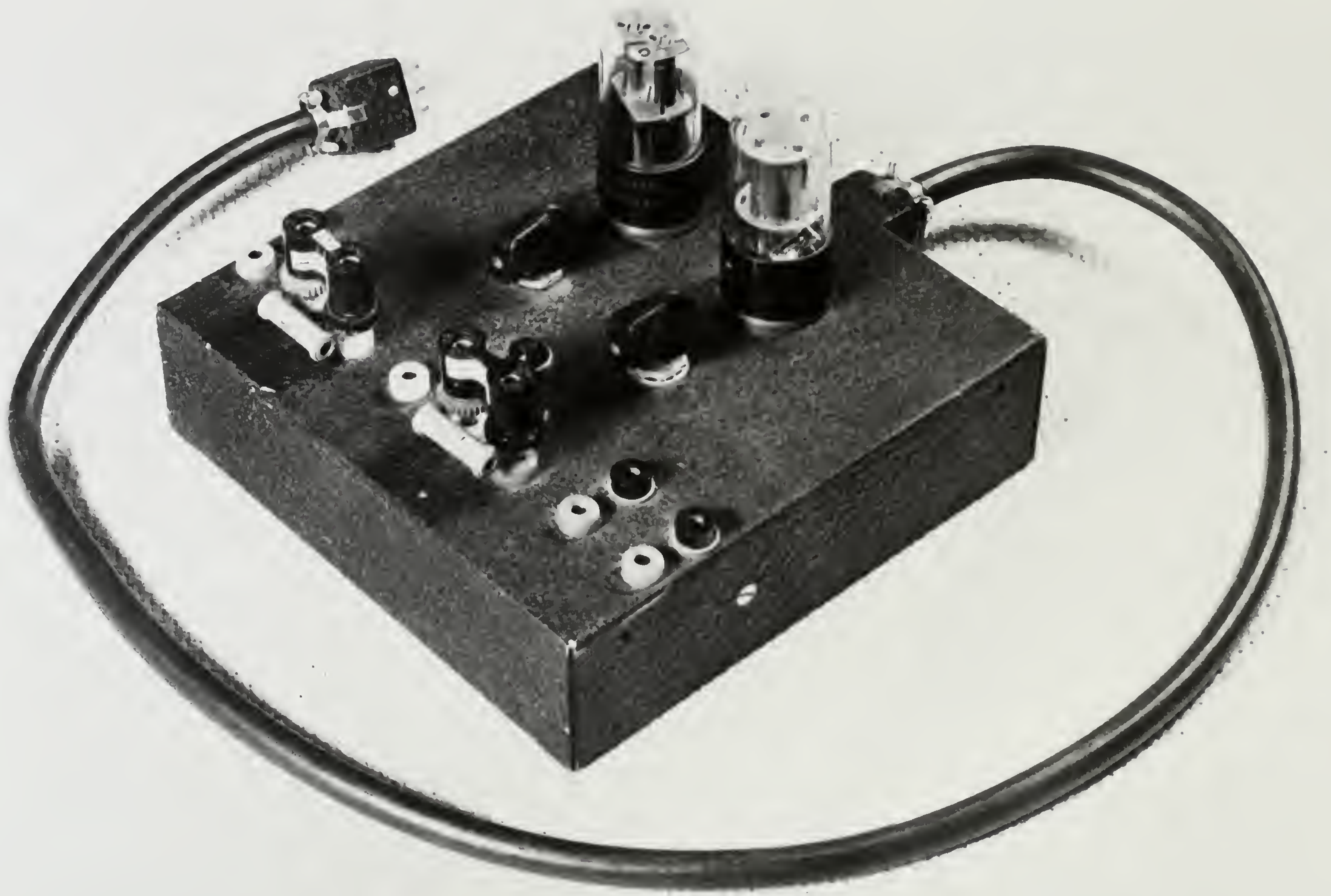


Figure 2 D. C. Amplifier Chassis

Figure 3

D.C. Amplifier Set Up as a Multiplier.



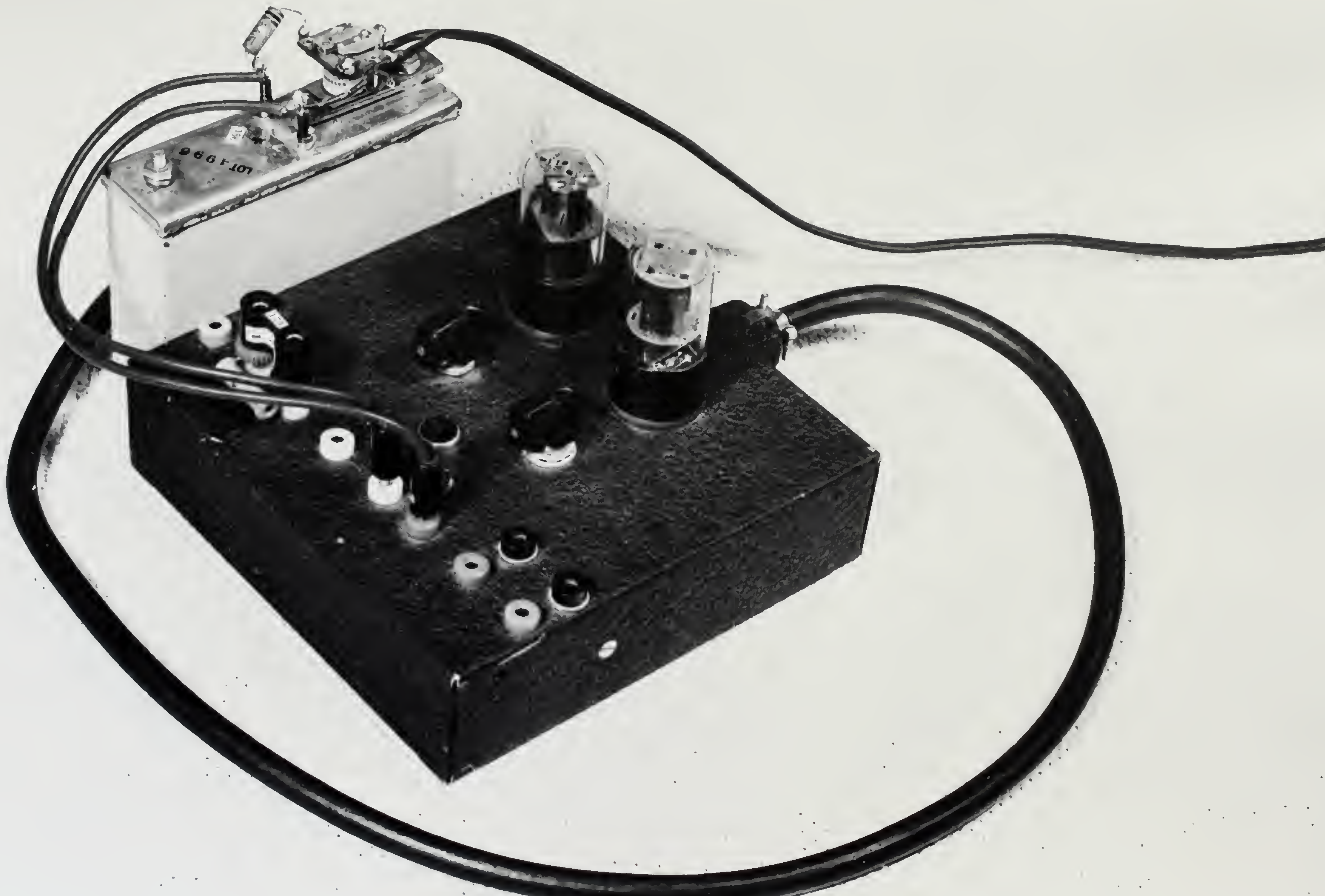


Figure 4 D. C. Amplifier Set Up as an Integrator

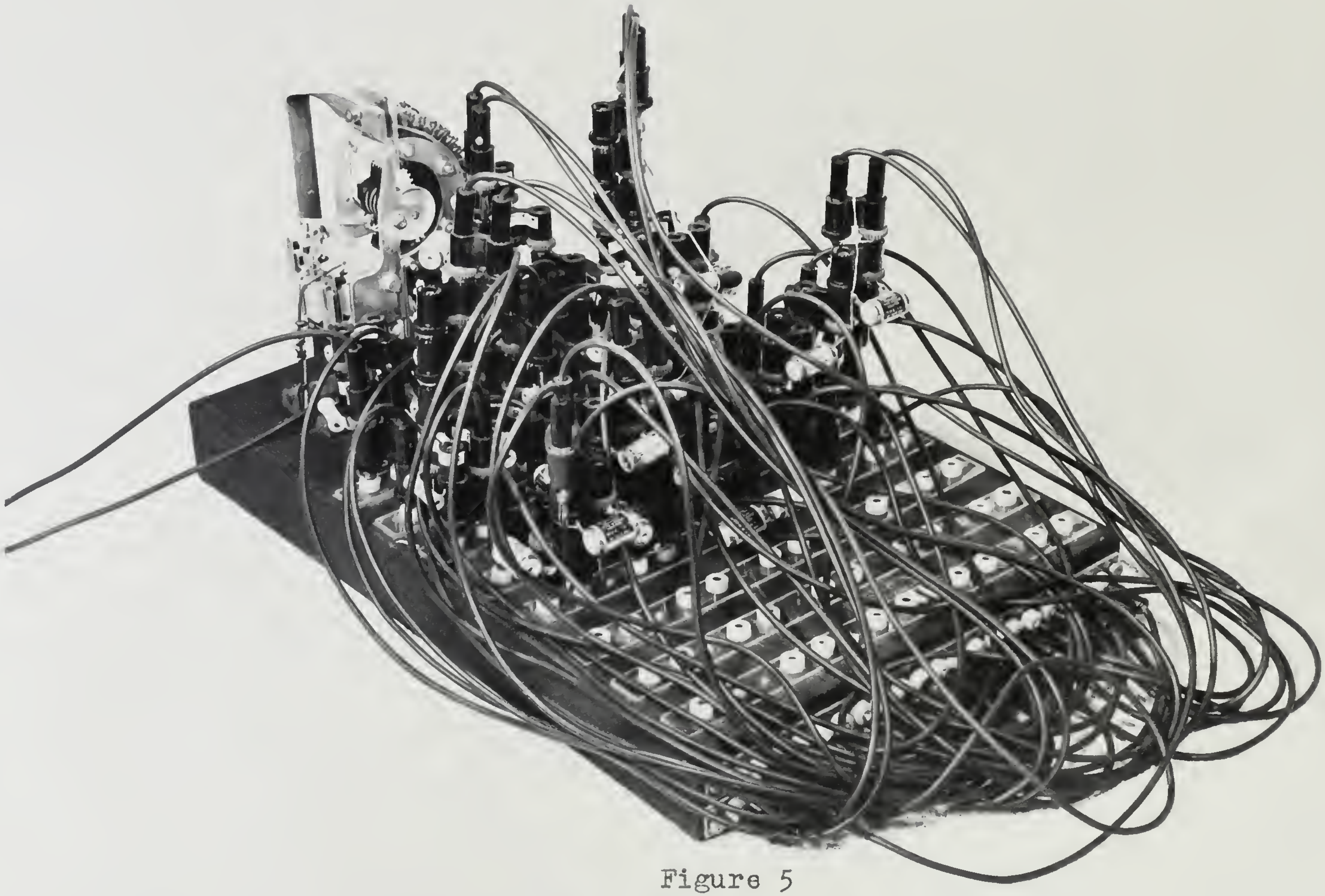


Figure 5

Stepping Relay and Resistor Panel Unit.

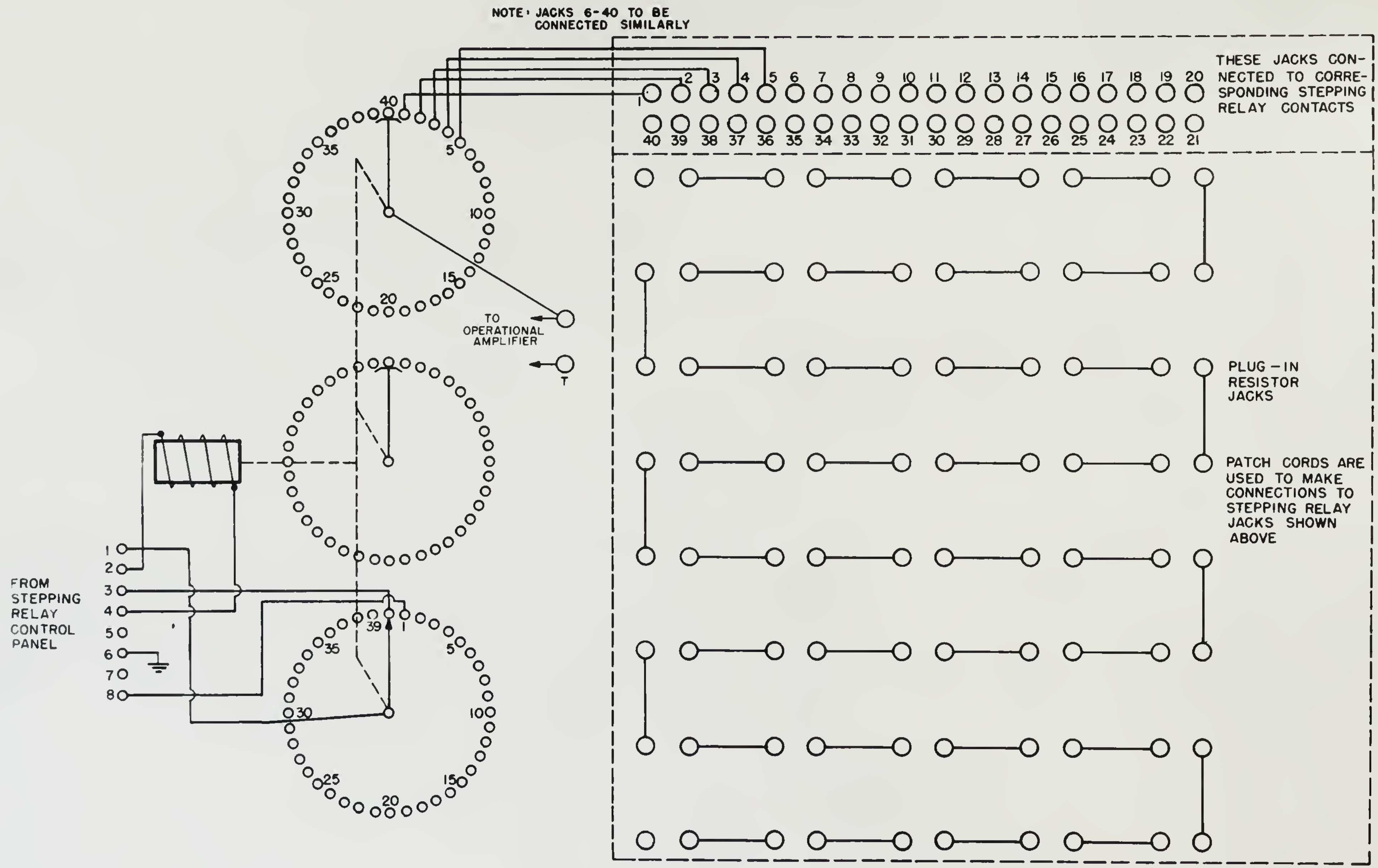
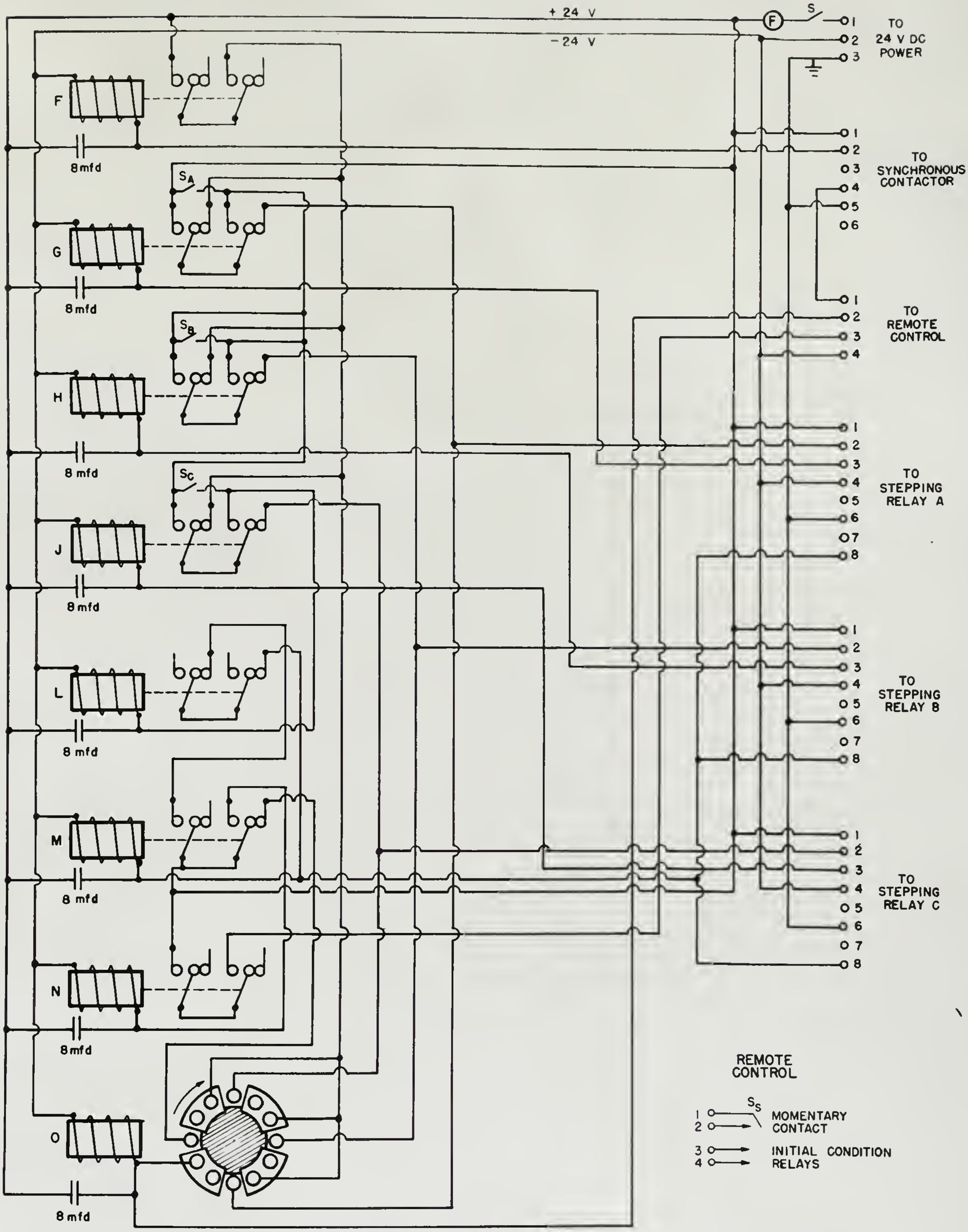


Figure 6.
STEPPING RELAY AND PLUG-IN RESISTOR CIRCUIT

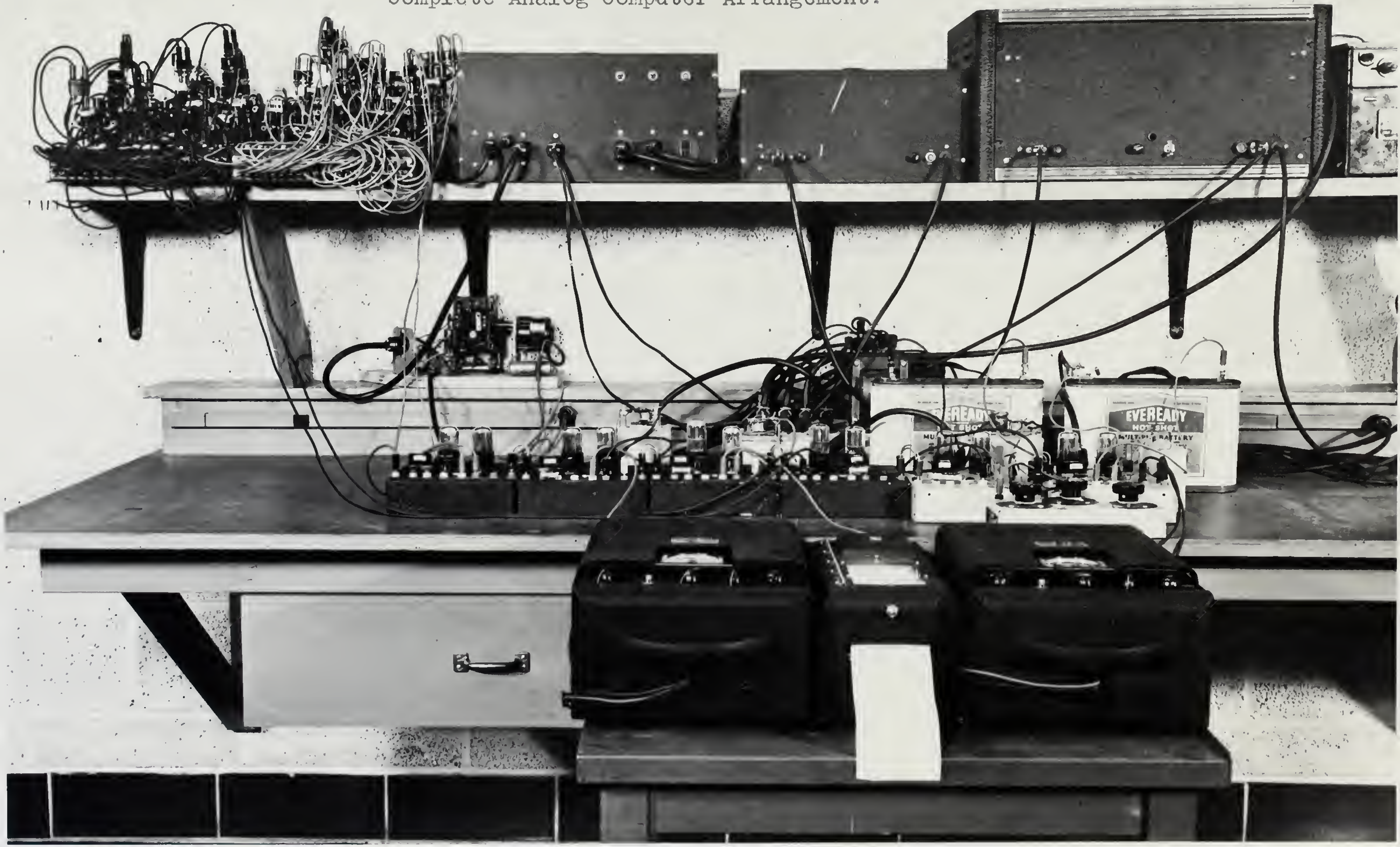


STEPPING RELAY CONTROL CIRCUIT

Figure 7

Figure 8.

Complete Analog Computer Arrangement.



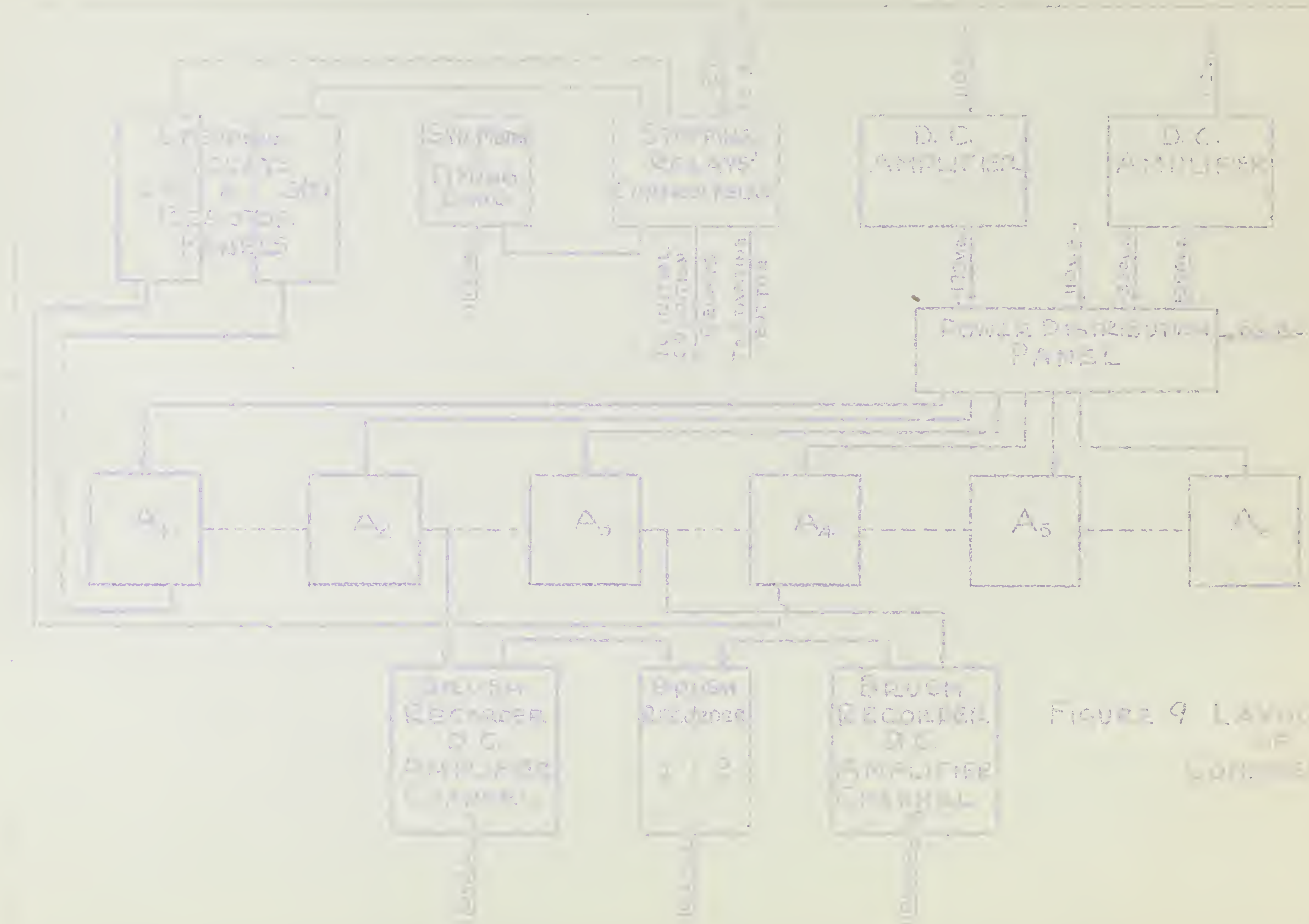


FIGURE 9 LAYOUT OF

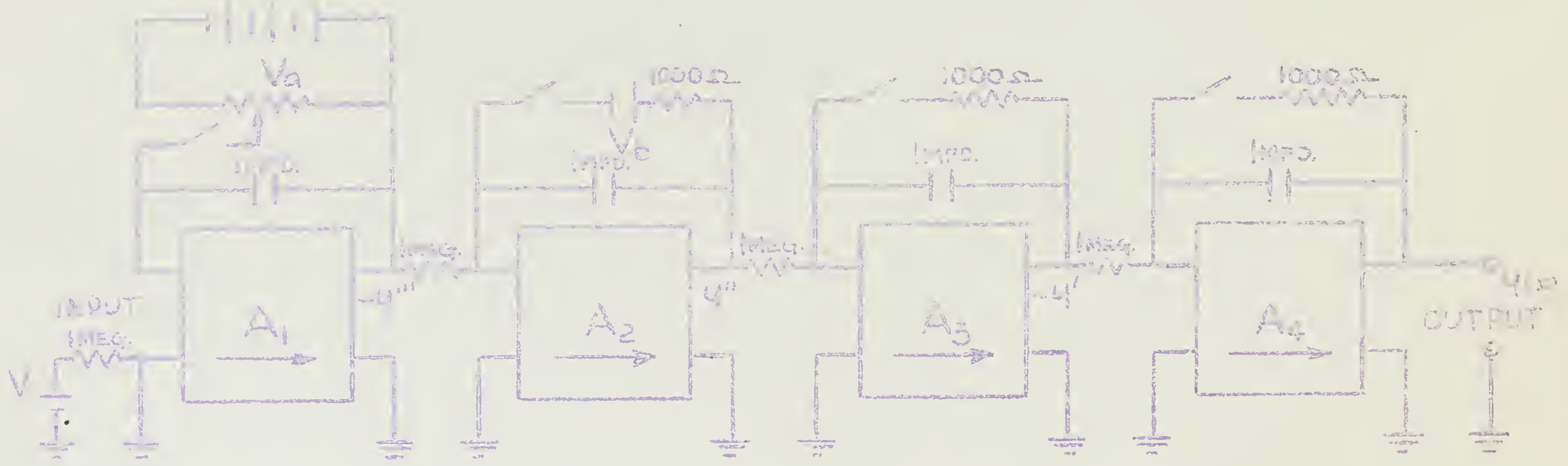


FIGURE 10 ANALOG COMPUTER CIRCUIT FOR $EI\frac{d^4y}{dx^4} = w(x)$

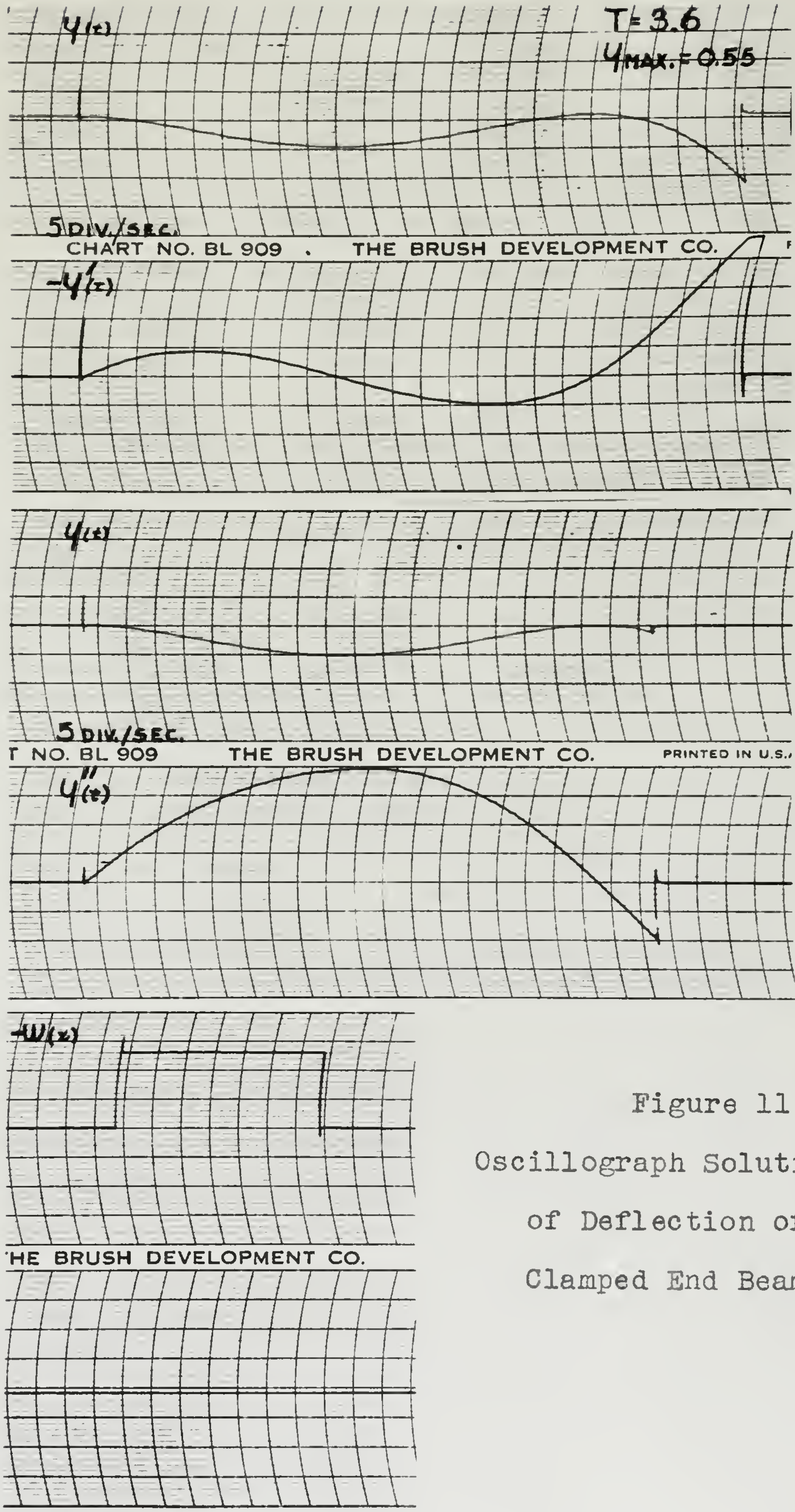


Figure 11

Oscillograph Solutions of Deflection of Clamped End Beam.

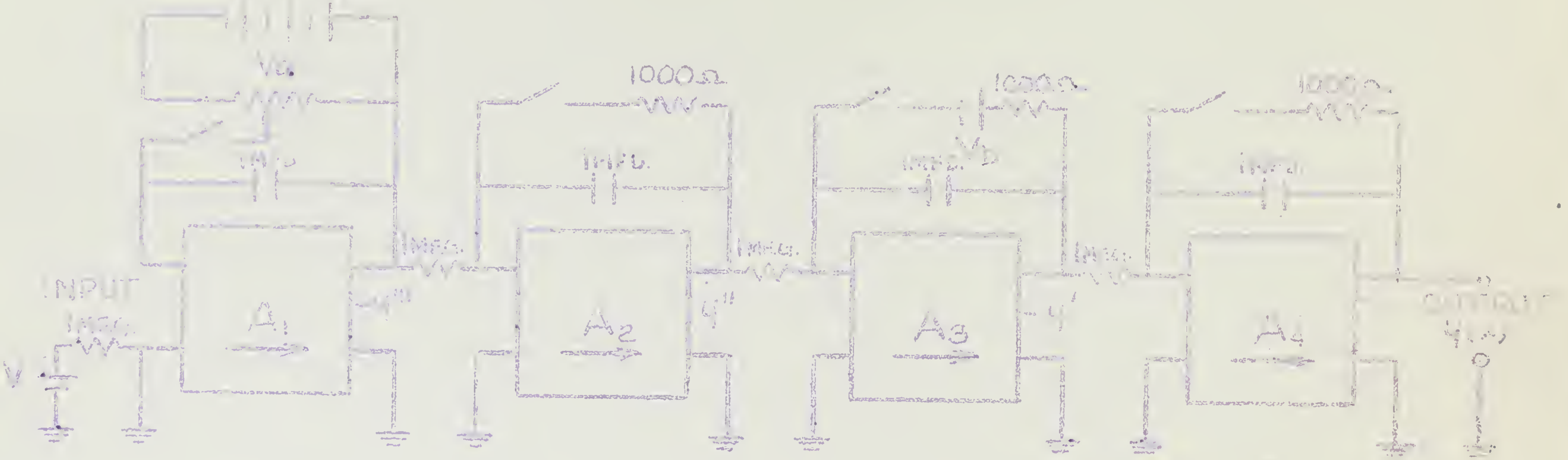
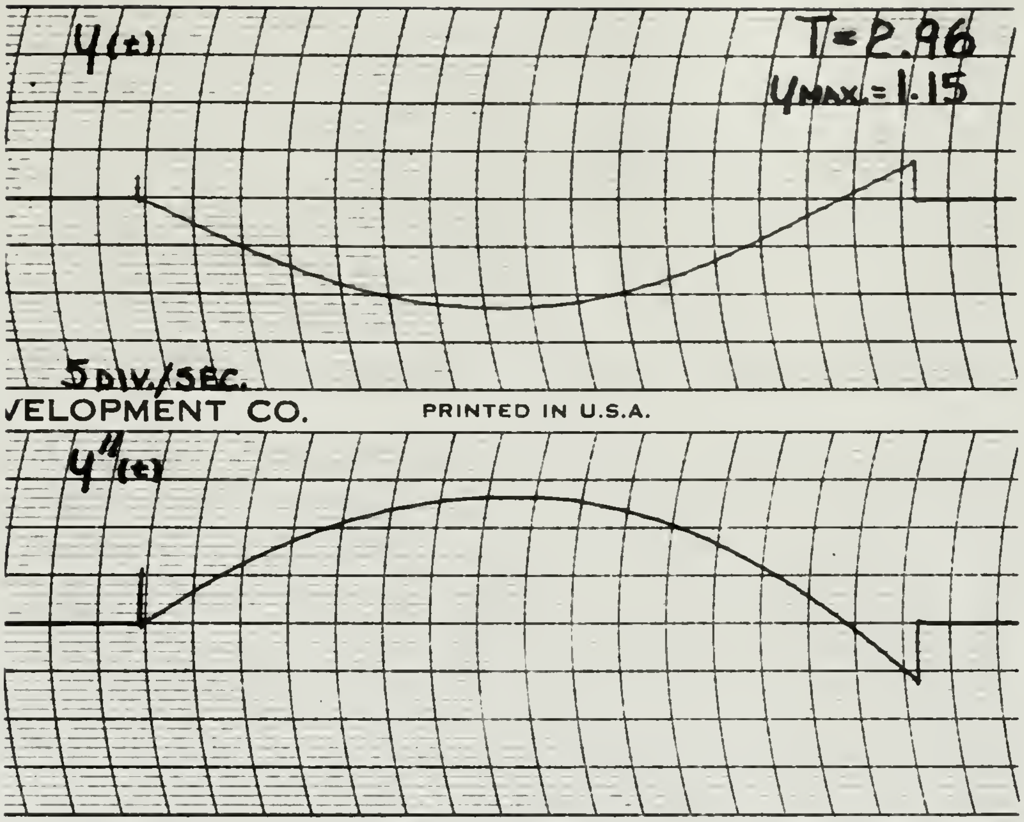


FIGURE 12 ANALOG COMPUTER CIRCUIT FOR $EI\frac{d^4y}{dx^4} = w(x)$



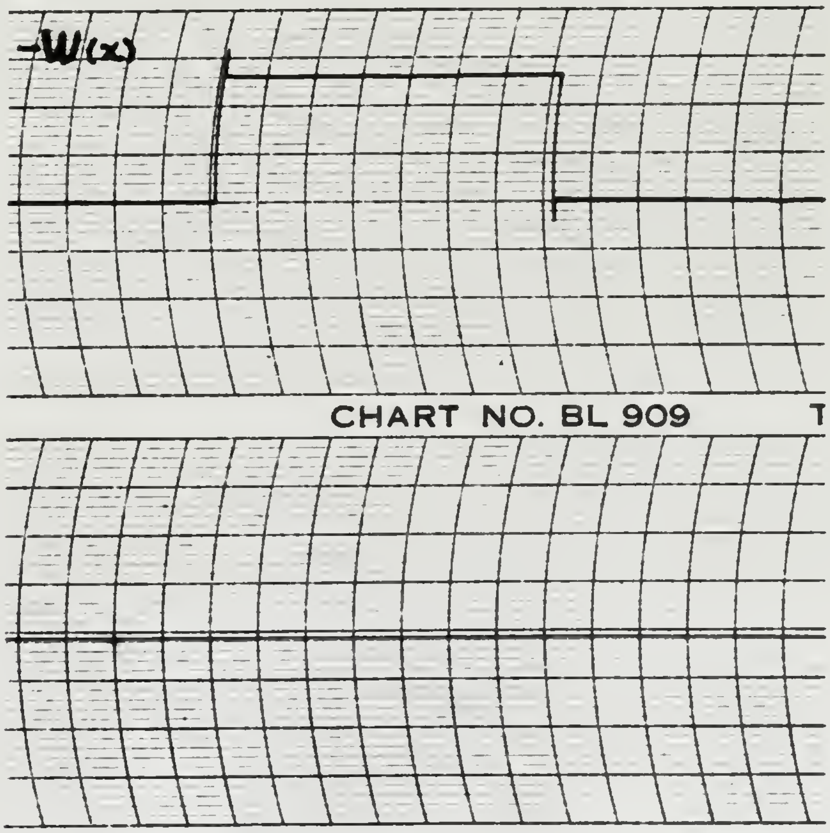


Figure 13

Oscillograph Solution of Deflection of Hinged End Beam.

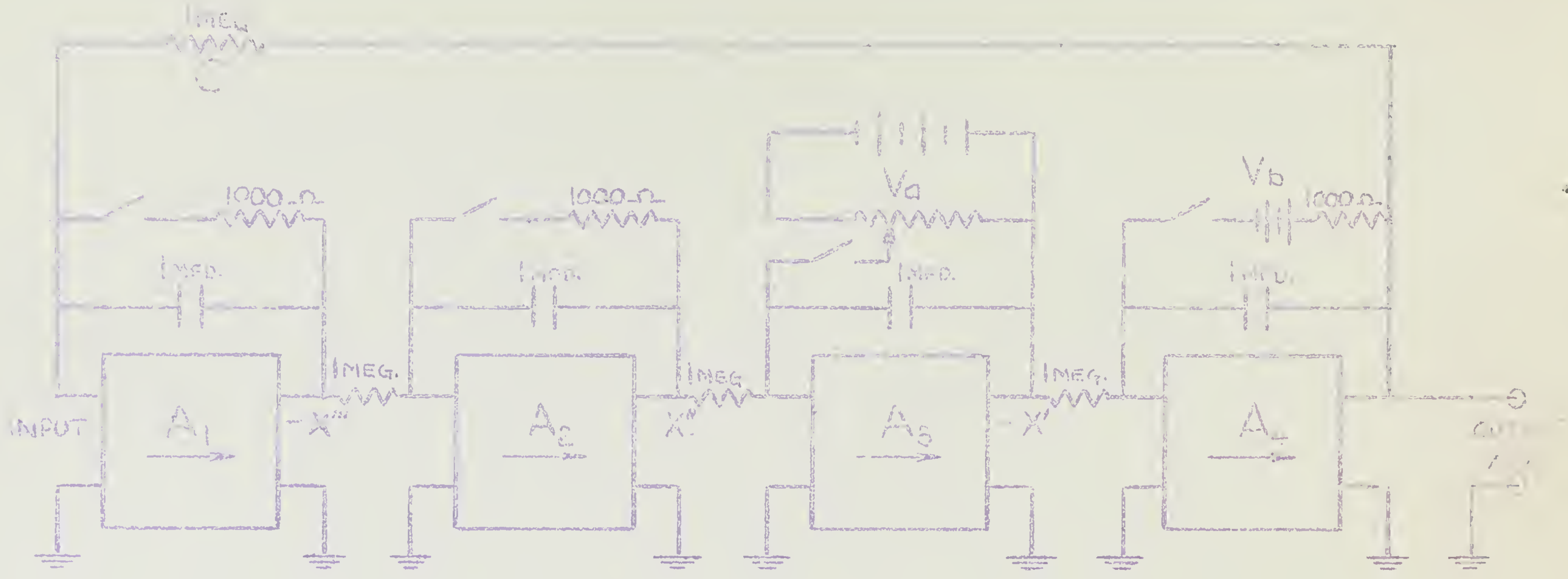


FIGURE 14 ANALOG COMPUTER CIRCUIT FOR $C\frac{d^4X}{dt^4} - X = 0$

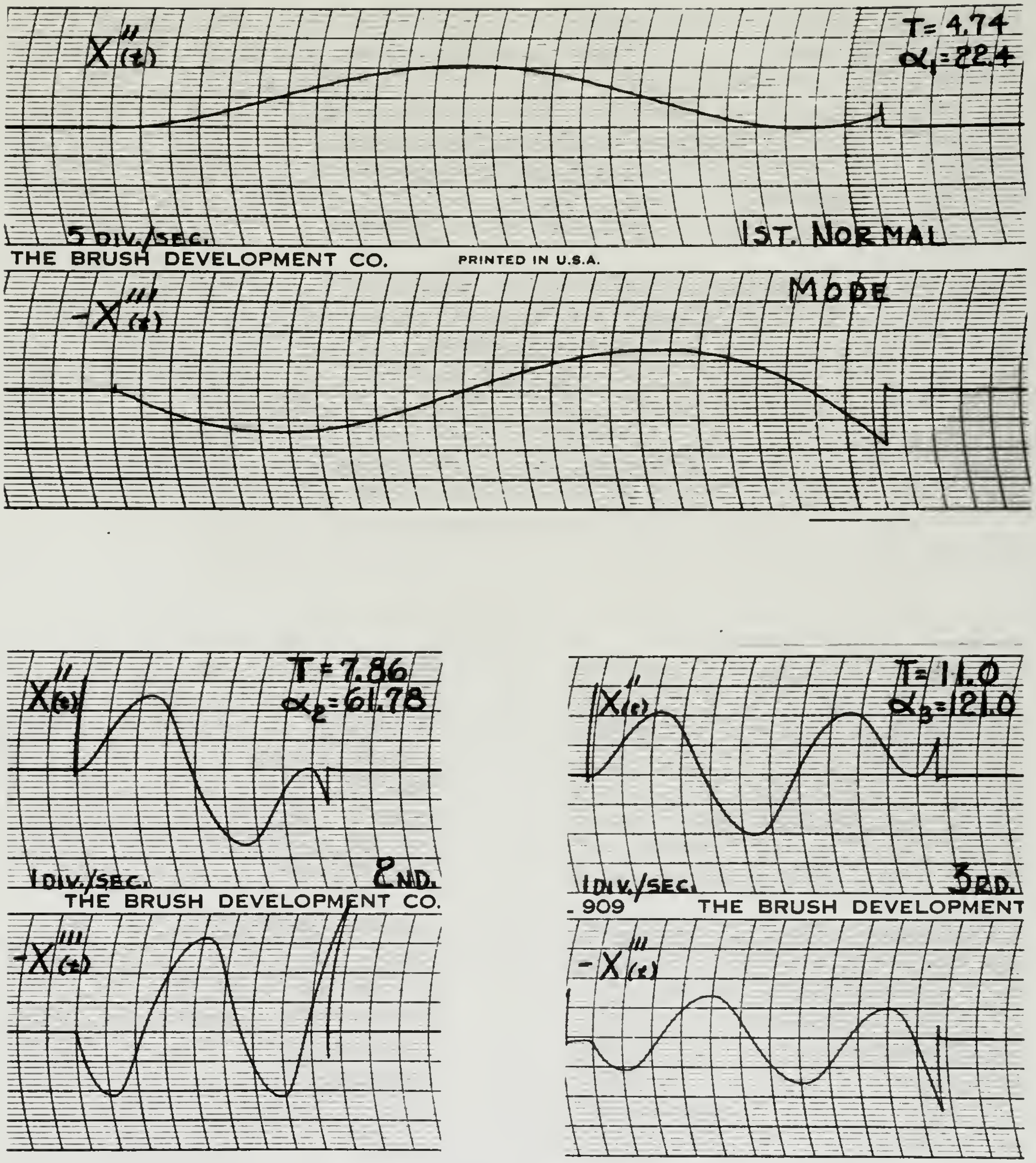


Figure 15

Oscillograph Solutions of First Three Normal Modes,

Uniform Free-free Beam.

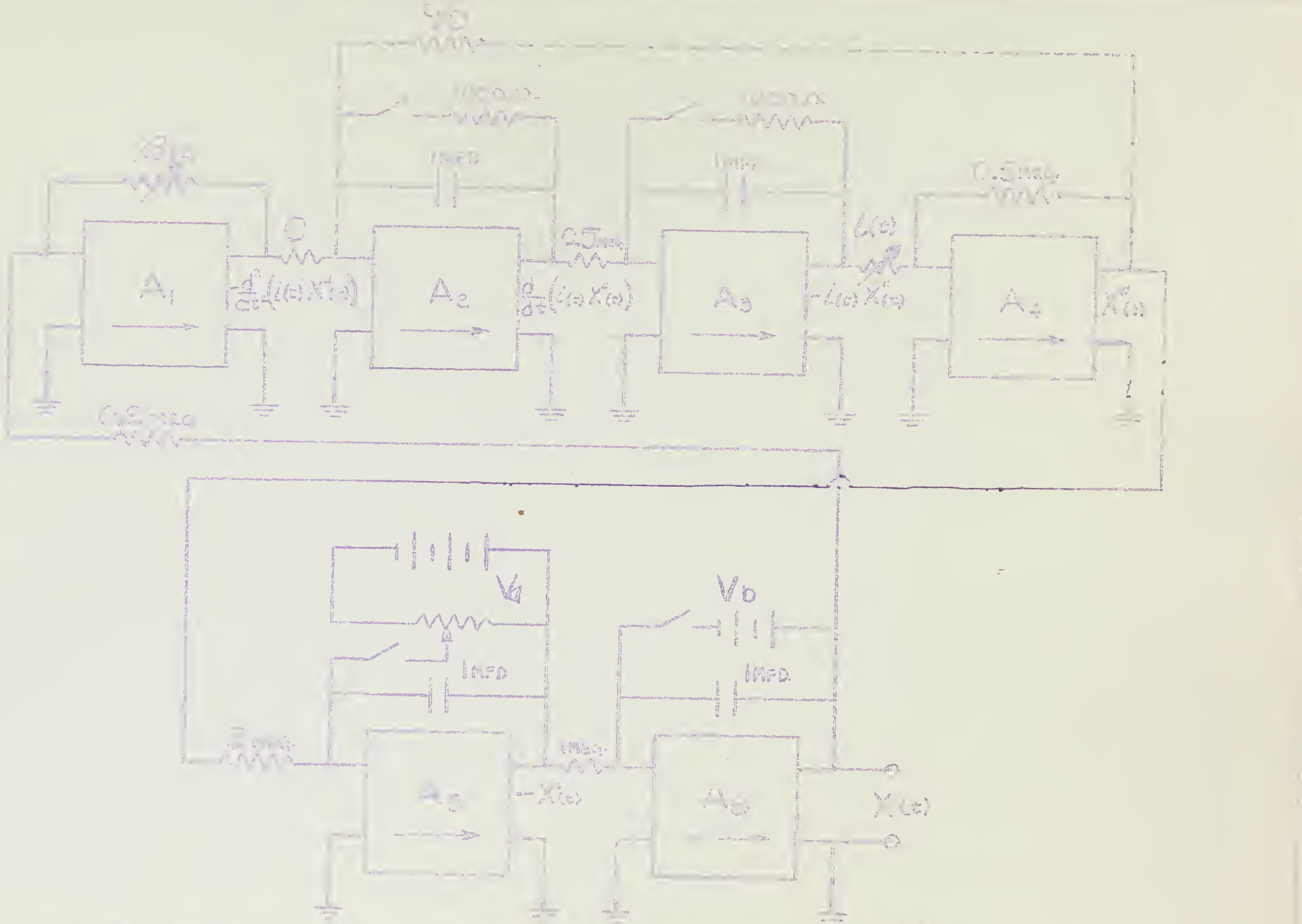


FIGURE 15 ANALOG COMPUTER CIRCUIT FOR $C\left[\frac{d^2}{dt^2}\left(\iota(t)\frac{d^2X}{dt^2}\right)\right] - \beta(t)X = 0$

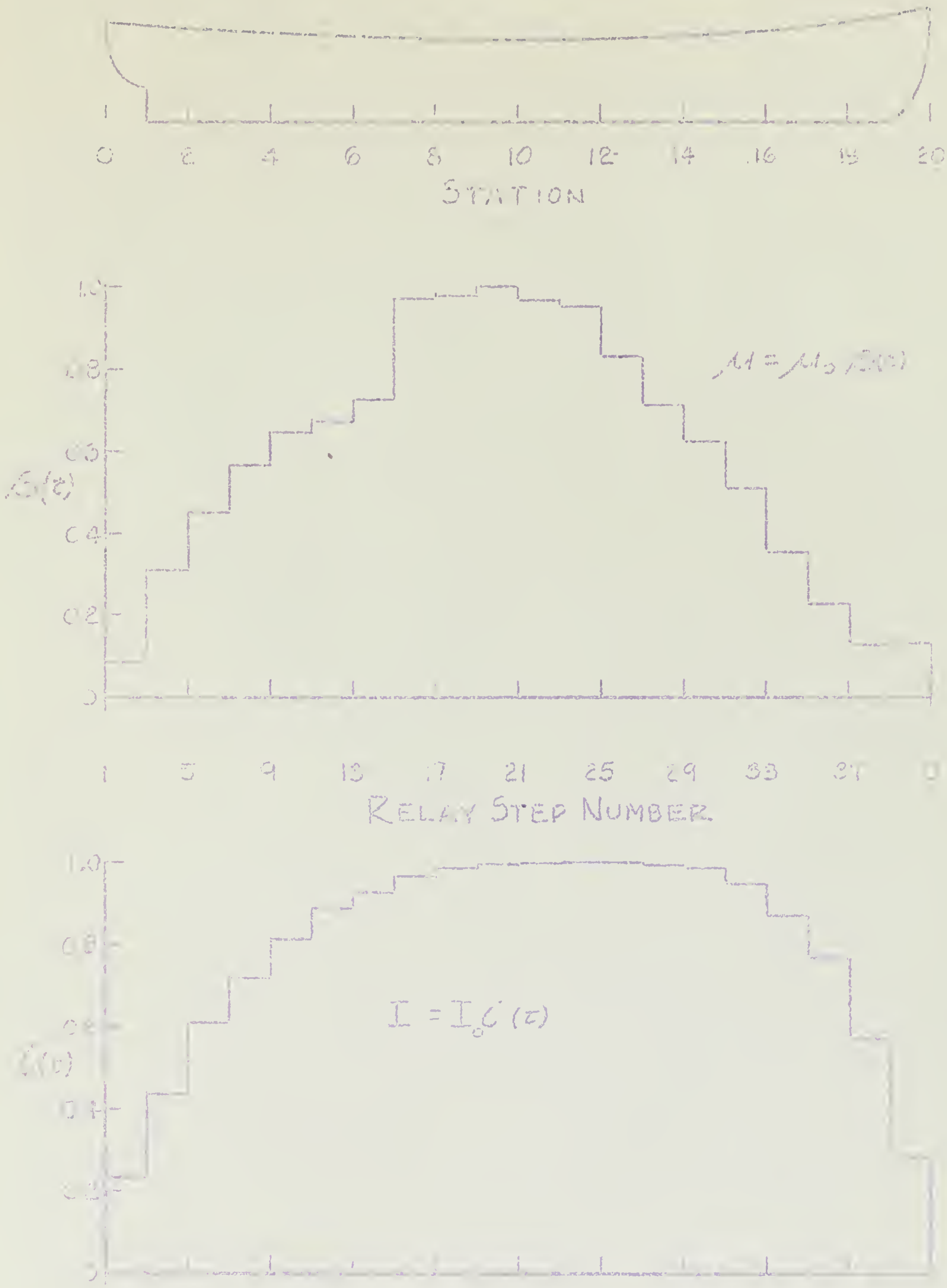
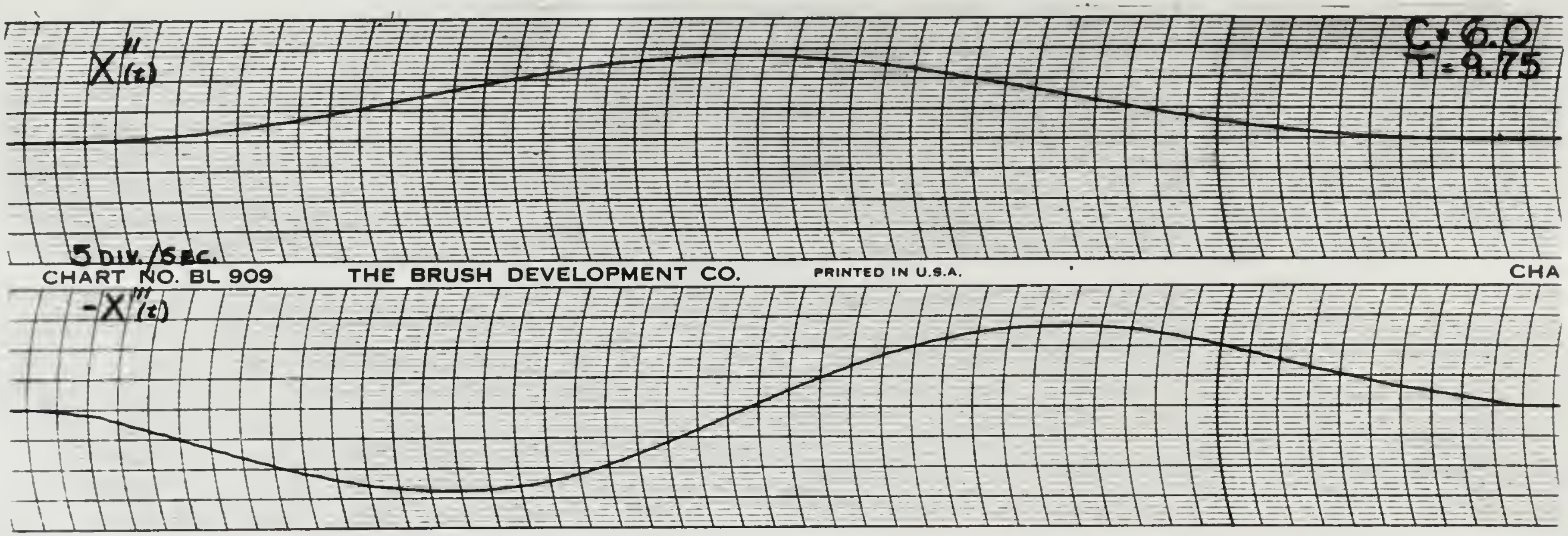


FIGURE 17 DISTRIBUTION OF VIRTUAL MASS AND MOMENT OF INERTIA [illegible]
AS INTRODUCED INTO COMPUTER



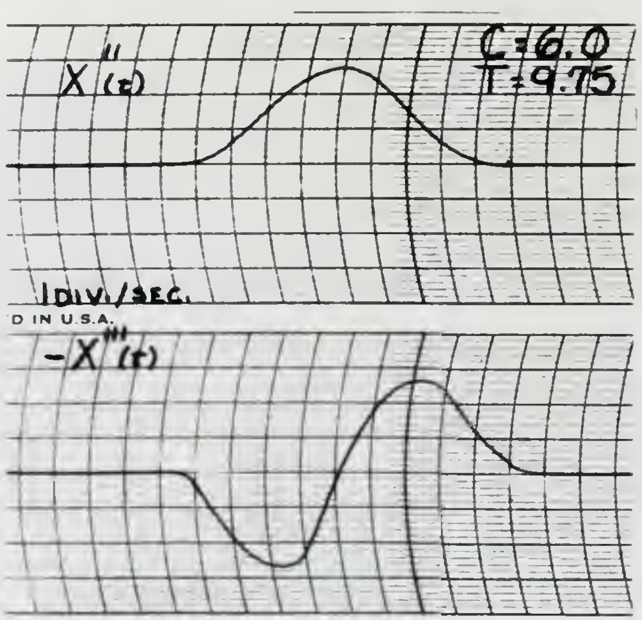
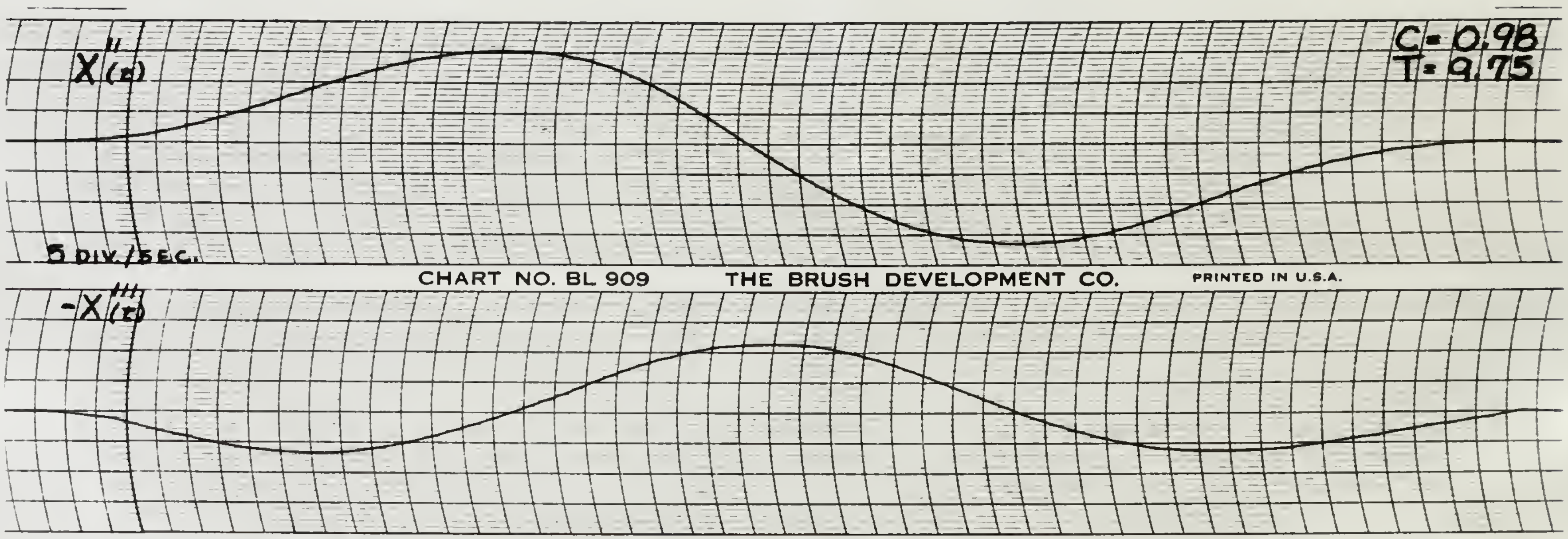


Figure 18

Oscillograph Solution of First Normal Mode

APA 87, Bending Deflections Only.



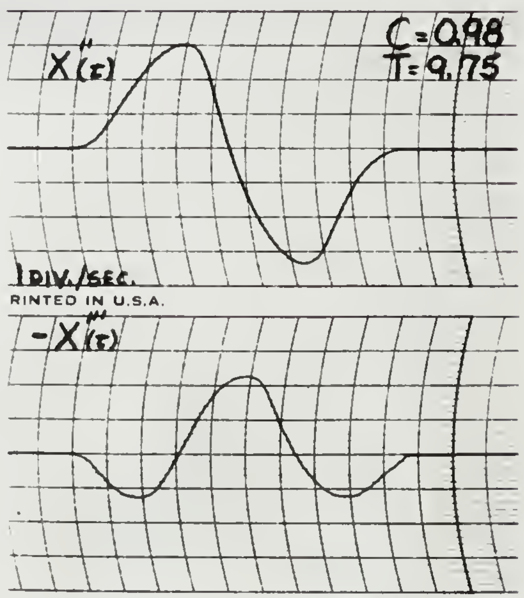


Figure 19

Oscillograph Solutions of Second Normal Mode

APA 87, Bending Deflections Only

FIGURE 20

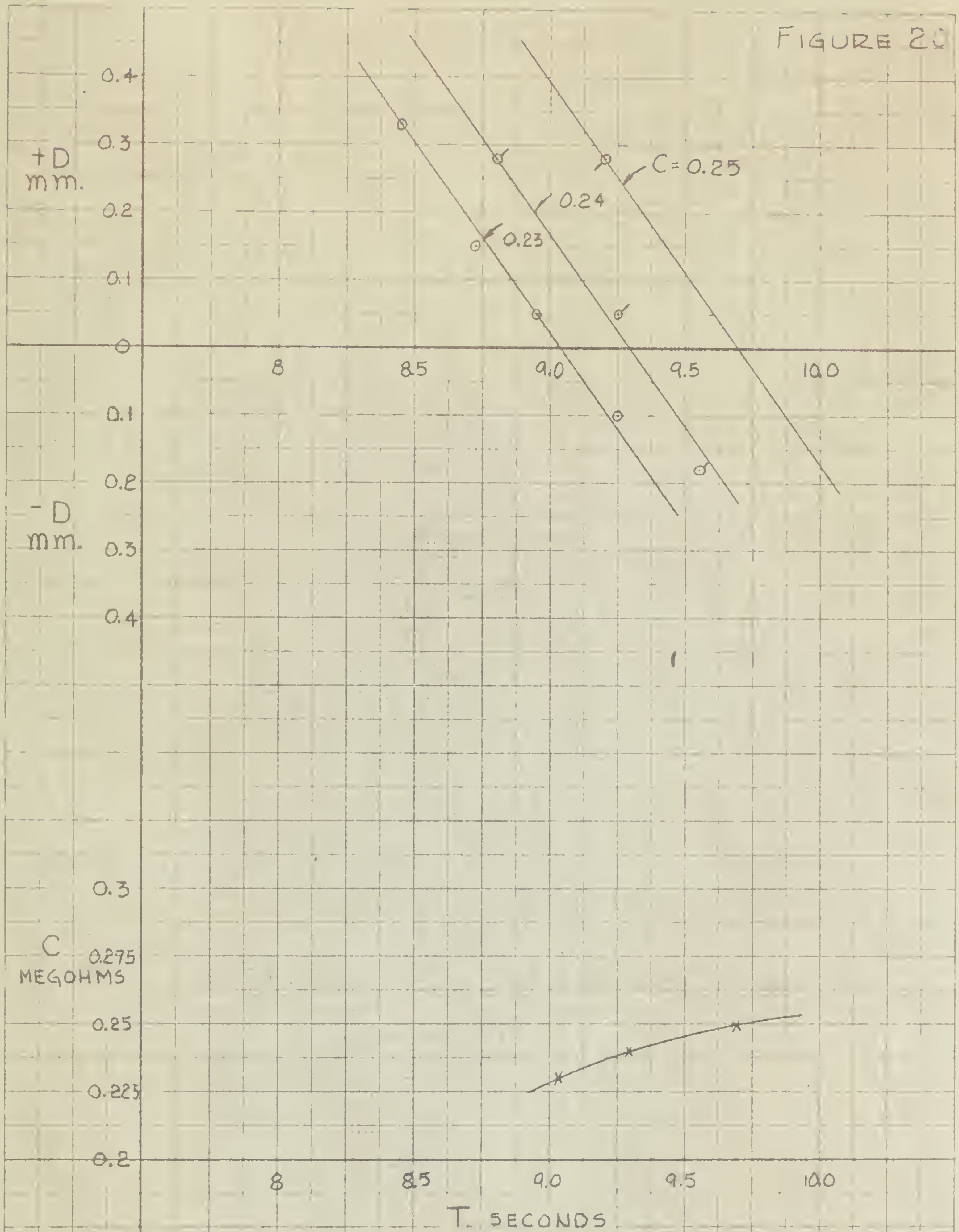


FIGURE 20

GRAPHICAL SOLUTION FROM OSCILLOGRAPH RECORDS OF THIRD MODE, APA 87, BENDING DEFLECTIONS ONLY.

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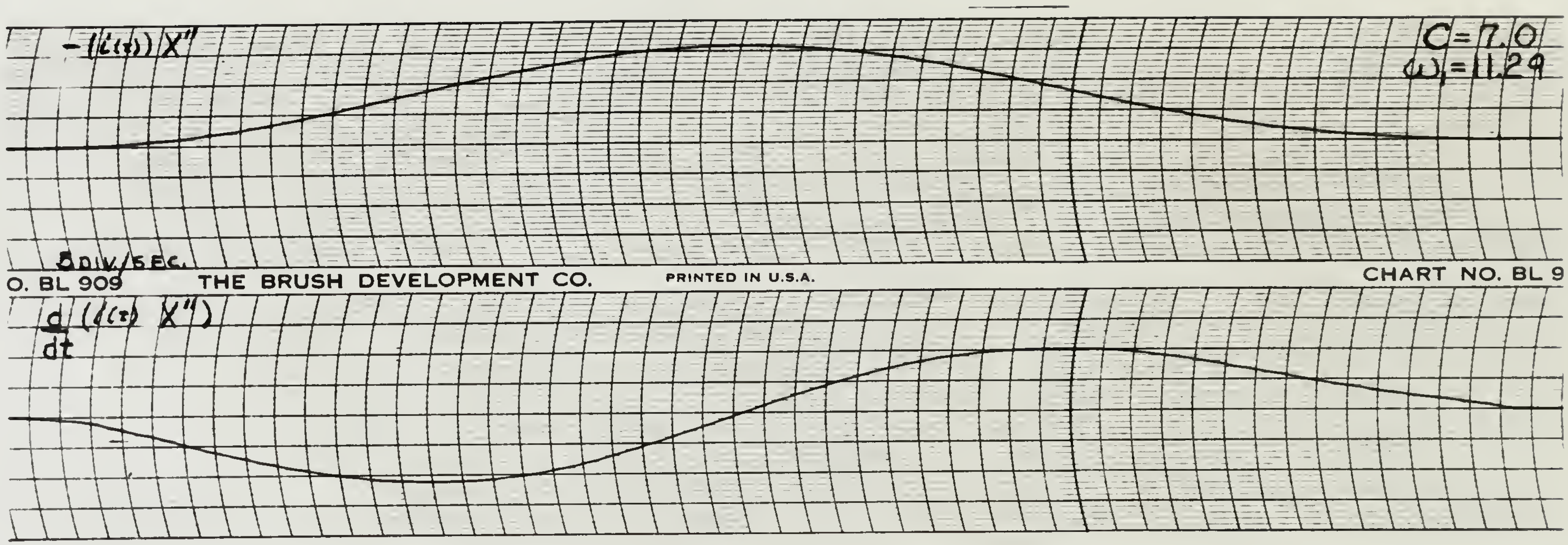


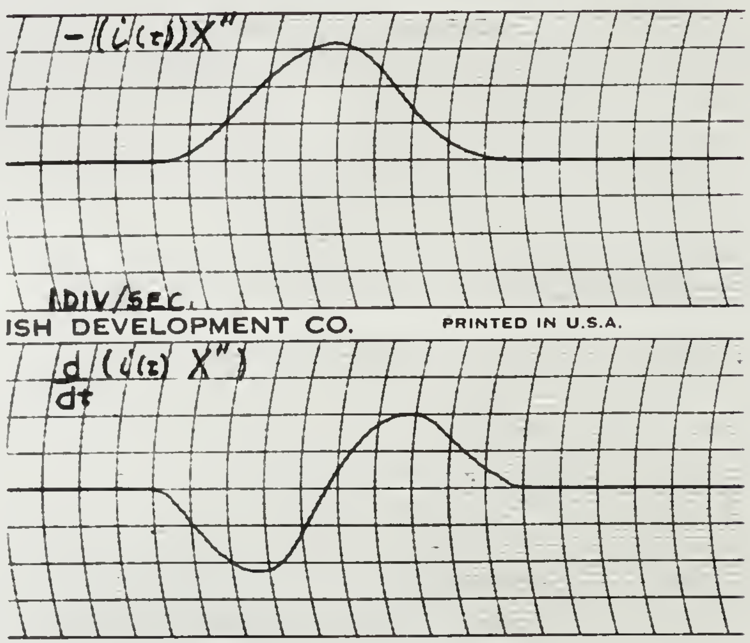
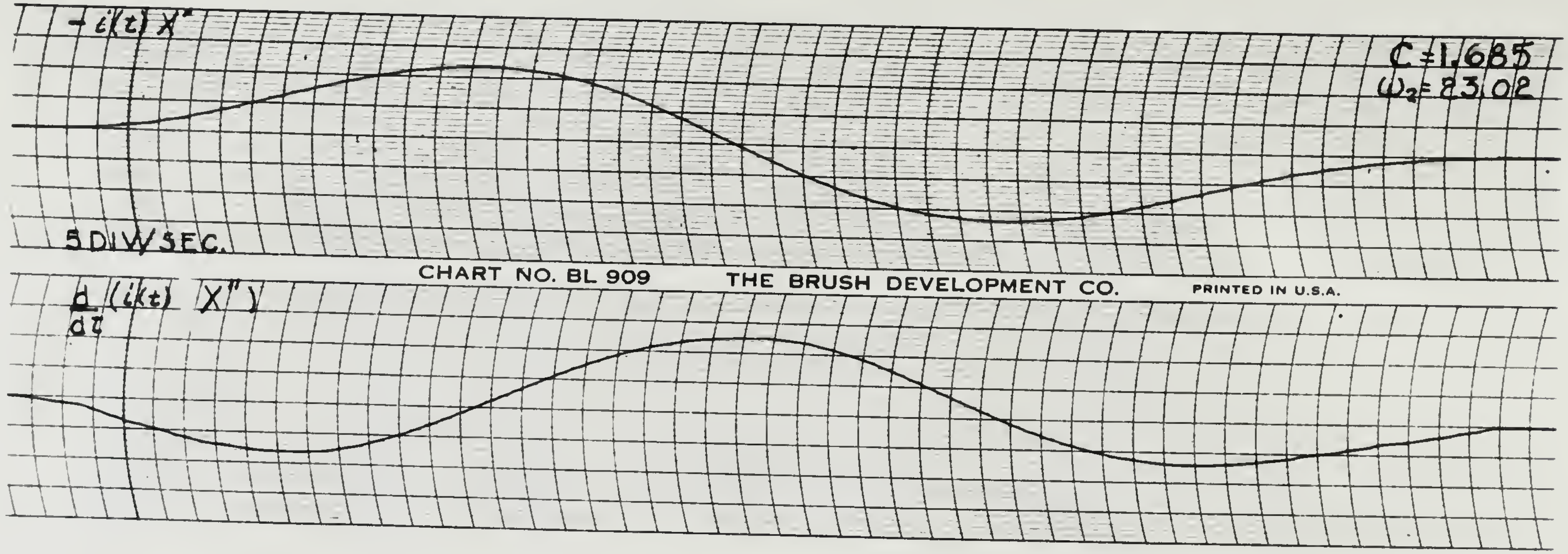


Figure 21

Oscillograph Solution of First Normal Mode, APA 87, Bending and Shear Deflections and Rotary Inertia.



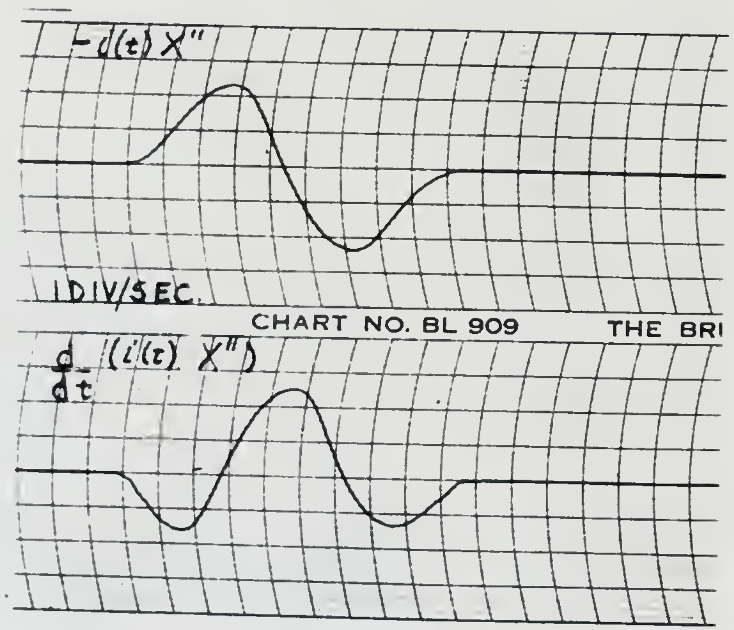
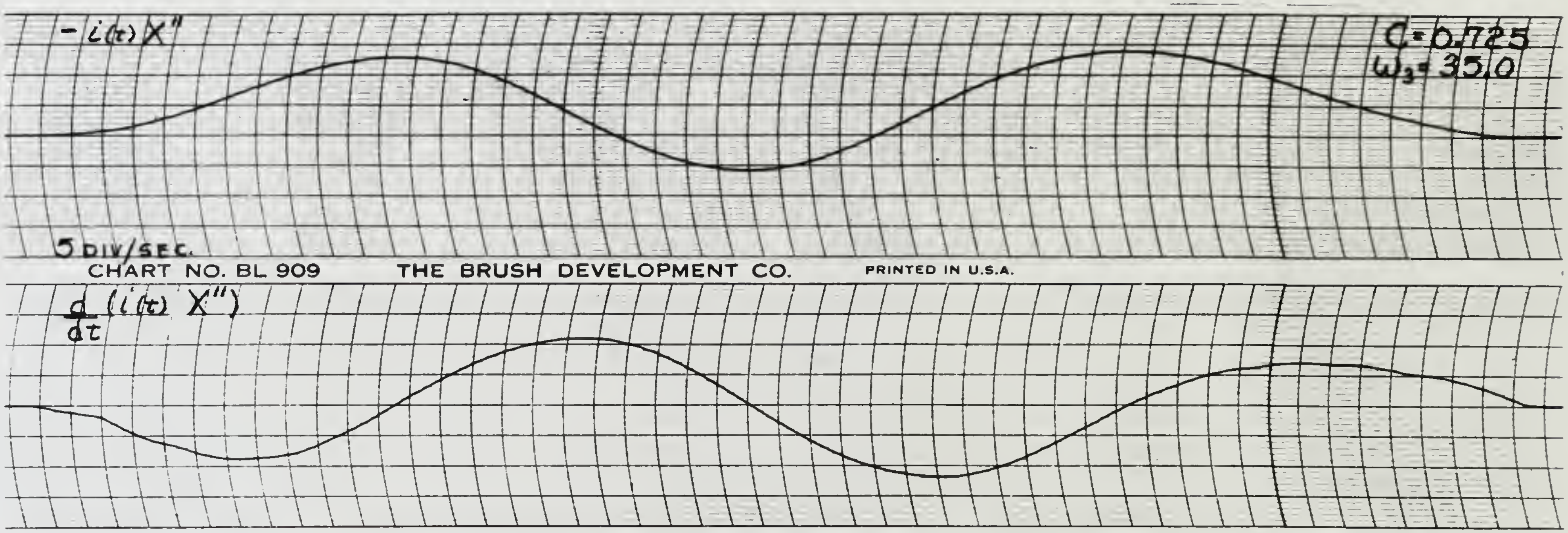


Figure 22

Oscillograph Solution of Second Normal Mode, APA 87, Bending and Shear Deflections and Rotary Inertia.



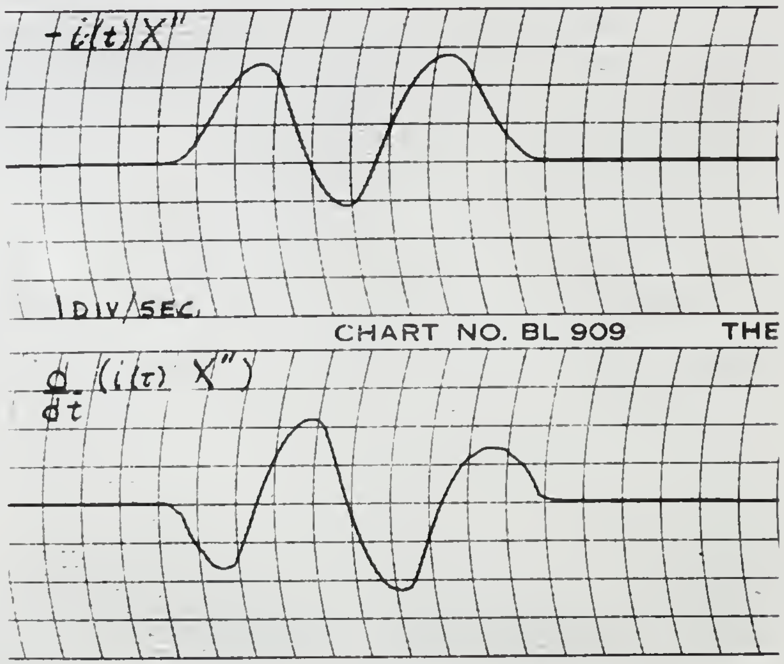


Figure 23

Oscillograph Solution of Third Normal Mode, AFA 37, Bending and Shear Deflections and Rotary Inertia.

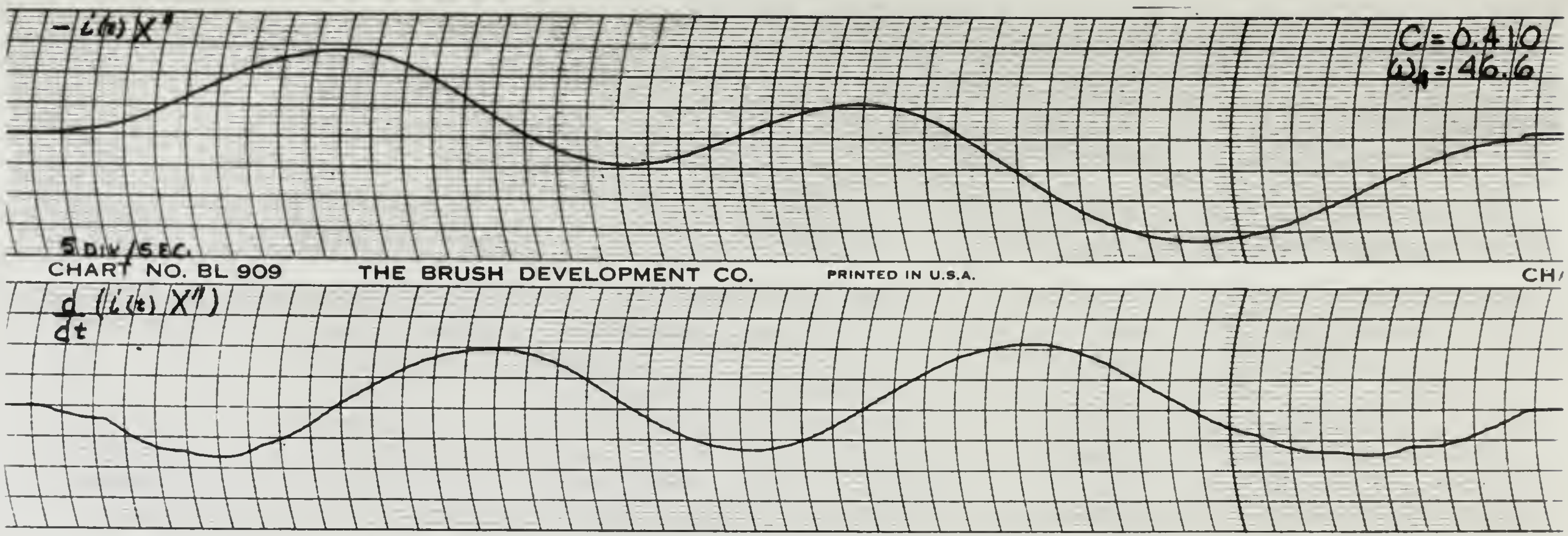


Figure 24

Oscillograph Solution of Fourth Normal Mode, APA 87,

Bending and Shear Deflections and Rotary Inertia.